LEARNING, TEACHING AND APPLYING GEOMETRY AND HANDLING DATA
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PBMa121: LEARNING, TEACHING AND APPLYING GEOMETRY AND HANDLING DATA

COURSE DESCRIPTION

Geometry is a critical component of mathematics education because student teachers are required to relate concepts from geometry to geometric phenomena. It provides the necessary mathematical tools for complex reasoning and solving problems in the sciences, technology, engineering, and many skilled trades and professions.

Handling Data also provides tools for describing variability in data and for making informed decisions. This course is designed to develop and consolidate the basic mathematical knowledge and skills in the domain of Geometry and Handling Data taking into account uses of mathematics in different local contexts as well as exploring learners’ misconceptions and difficulties in these domains. Student teachers will be required to demonstrate good understanding of all the areas covered by the senior high school core mathematics, especially areas where the chief examiners’ reports have highlighted as difficult. There is the need to auditing of subject knowledge to establish and address student teachers’ learning needs, perceptions and misconceptions in Geometry and Handling Data.

These areas include, but not limited to, bearing – representing the given information on a correct diagram; circle geometry and its applications; mensuration of plane and three dimensional shapes; drawing required diagrams correctly; geometrical construction; geometry and basic trigonometry with applications; representation of information in diagrams; congruence and similarities; finding angles and distances; global mathematics, introductory statistics and probability; cumulative frequency curve; drawing and reading from graphs; reading and answering questions from graphs; probability: meaning and application in real-life situations.

The student teacher will also be required to demonstrate the ability to identify how their own individual characteristics (culture, ethnicity, religion, family constellation, socio-economic background, dis/ability, etc.). Differentiated approach to teaching will be used to ensure that student teachers will be supported in the area of Geometry and Handling Data. The course will focus on mathematical content on one hand and the strategies and learning experiences in doing mathematics on the other hand. These will be combined to form an integrated instructional approach that addresses the course learning outcomes. The instructional strategies will pay attention to all learners, especially girls and students with Special Education Needs. The course will be assessed using a variety of assessments methods including coursework (assignments, quizzes, project works, and portfolio entries with presentation) and end of semester examination to provide a comprehensive outlook of student teachers’ competencies and skills. References are made to the following (NTS, 2b, 2f, 3j, 3m) and (NTECF p.30, p.39).
<table>
<thead>
<tr>
<th>Outcomes</th>
<th>Indicators</th>
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<tbody>
<tr>
<td>On successful completion of the course, student-teachers will be able to:</td>
<td>1. Select and use the most appropriate mathematical method(s) or heuristics in carrying out tasks/exercises/problems in Geometry and Handling data within the basic education mathematics foundation list.</td>
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<tr>
<td>1. Demonstrate deep understanding of key mathematical concepts in Geometry and Handling Data content domains in the basic school mathematics curriculum (professional values, knowledge &amp; practice) (NTS, 2b)</td>
<td>1.2 Make connections between mathematical concepts in Geometry and Handling Data content domains and applying them to solve real-life problems.</td>
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<td>1.3 Identify and resolve mathematics related learning difficulties within Geometry and Handling Data content domains such as inability to visualise geometrical shapes.</td>
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<td>2. Use manipulatives and other TLMs including ICT in a variety of ways in learning mathematics concepts in Geometry and Handling data (practical skills, digital literacy, problem solving) (NTS, 3j);</td>
<td>2.1 Use manipulatives and other TLMs in developing Geometry and Handling data concepts.</td>
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<td></td>
<td>2.2 Use ICT as a tool in developing Geometry and Handling data concepts. E.g. Geometer Sketchpad, Geogebra</td>
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<td>2.3 Use drawing tools to conduct geometrical investigations emphasising visualization, pattern recognitions and conjecturing.</td>
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<td></td>
<td>2.4 Solve mathematics problems using manipulatives and/or technology related strategies in a variety of ways.</td>
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<td>3. Demonstrate value as well as respect equity and inclusivity as well as core skills in the mathematics classroom (knowledge)(NTS, 2f)</td>
<td>3.1 Both tutors and student-teachers do individual reflection on their knowledge of Geometry and Handling Data.</td>
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<td>3.2 Identify and reflect on core skills applied in the mathematics classroom.</td>
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<td>3.3 Appreciate the contributions of, and supports, colleagues in the mathematics classroom.</td>
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<td>3.4 Cooperate with colleagues in carrying out mathematics tasks in Geometry and Handling Data.</td>
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<td>3.5 Engage in reflective thinking about how mathematics was taught in student-basic and high school days.</td>
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<td>4.2 Identify appropriate TLMs for teaching topics in Geometry and Handling data</td>
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<td>4.3 Identify and use manipulates in Geometry and Handling data lessons</td>
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## COURSE CONTENT

<table>
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<tr>
<th>Uni</th>
<th>Topics</th>
<th>Subtopics</th>
<th>Teaching and learning activities to achieve learning outcomes</th>
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<tr>
<td>1</td>
<td>Plane Geometry (Patterns in shape): <em>Learning, teaching and applying</em></td>
<td>Angles at a point, angles and parallel lines, angles and triangles. Properties of triangles, quadrilaterals and polygons. Learning about 3-Dimensional shapes: comparing polyhedral, forming 3-Dimensional shapes, Learning about 2-Dimensional shapes: polygons ($n \geq 3$), tessellations and applying these to the teaching of the JHS Mathematics curriculum, Congruence and similarities (teaching symmetry, congruence and similar shapes,)</td>
<td>Use tutor-led and student-led presentations on the teaching and learning of patterns in shape Use investigations to explore perceptions, properties and application of angles and polygons. Group discussion of the application of 2D and 3D shapes in real situations, Use shapes to explore properties of symmetry and congruency in the basic school mathematics curriculum, Explore through problem-solving application of congruence and symmetry.</td>
</tr>
<tr>
<td>2</td>
<td>Geometrical Constructions: <em>Learning, teaching and applying</em></td>
<td>Teaching measurement of a line, bisection of a line and angles and construction of basic angles ($60^\circ$, $90^\circ$, $30^\circ$, $15^\circ$, $45^\circ$). Teaching construction of other angles (eg. $75^\circ = 45^\circ + 30^\circ$, $105^\circ = 90^\circ + 15^\circ$). Teaching construction of triangles, quadrilaterals and loci and their applications in the basic school mathematics curriculum.</td>
<td>Use sets of construction tools to construct given shapes and angles. Use verbal exposition to identify common misconceptions from students’ work in construction. Use group work to explore the relationships between the various angles that can be constructed</td>
</tr>
<tr>
<td>3</td>
<td>Basic trigonometry: <em>Learning, teaching and applying</em></td>
<td>Teaching and application of right-angled triangle, Pythagorean triples, trigonometry ratio (sine, cosine and tangent), trigonometry applications to real life</td>
<td>Tutor-led and student-led presentations on the application of trigonometric ratios. Using explorations to establish basic trigonometry ratios and their applications in the teaching of geometry.</td>
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<td>4</td>
<td>Vectors and Bearing:</td>
<td>Algebra of vectors, vector representation notation components</td>
<td>Using worksheets on bearing to explore the relationship between angles in bearing and back bearing</td>
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<tr>
<td>5</td>
<td>Mensuration: <em>Learning, teaching and applying</em></td>
<td>Teaching parts of a circle. Teaching measurement of length (arc length, radius, diameter, chord) Teaching area of a sector, area of segment, volume of cone, cylinder. Application of mensuration in real life problems</td>
<td>Project work – individual/group presentations on the application of circle concepts in real life situation</td>
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<td>6</td>
<td>Global Mathematics: <em>Learning, teaching and applying</em></td>
<td>The earth as a sphere, lengths on latitudes and longitudes.</td>
<td>Tutor led presentations on lengths on a sphere Using worksheets for practical investigation to distinguish between latitudes and longitudes</td>
</tr>
<tr>
<td>7</td>
<td>Introductory Statistics (Patterns in data): <em>Learning, teaching and applying</em></td>
<td>Teaching collection of data, measures of central tendencies, measures of dispersion, graphical representation (cumulative frequency)</td>
<td>Project work – individual/group presentations on data collection Discussion on establishing the relationship between the measures of central tendencies and measures of dispersion</td>
</tr>
<tr>
<td>8</td>
<td>Basic probability: <em>Learning, teaching and applying</em></td>
<td>Teaching basic concepts of probability: sample space, events, mutually exclusive and independent events. Applications to real life situation.</td>
<td>Interactive collaborative group work on probability. Exploring the concept of probability through experiments. Different ways of presenting probability through games in mathematics lessons.</td>
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</table>
**COURSE ASSESSMENT (EDUCATIVE ASSESSMENT: OF, FOR AND AS LEARNING)**

**COMPONENT 1: Examination**

**Summary of Assessment Method:**

Students should be summatively assessed by an examination linked to the themes listed below:

- knowledge, understanding and applications of the key mathematical concepts in Geometry and Handling Data within the basic school mathematics curriculum.
- use manipulatives and other TLMs including ICT in a variety of ways to establish Geometry and Handling Data concepts in the classroom
- how their mathematics history influences their views of mathematics in the realm of social context and how this affects their way of learning mathematics.
- relevant professional values and attitudes for teaching mathematics at basic school level

**Weighting: 40%**

**Assesses Learning Outcome(s):** CLO 1, 2, 3, 4; (NTS 2b, 2f, 3j)

**Component 2: Coursework 1**

**Summary of Assessment Method:**

Individual Assignments with Presentations: Student teachers may be asked to

- use ICT tools to conduct geometrical and statistical investigations emphasizing visualization, pattern recognitions, conjecturing etc. in a variety of ways.
- select the most appropriate mathematical method(s) or heuristics (i.e. using mental strategies, models, paper and pencil, etc.) in carrying out tasks / exercises / problems in Geometry and Handling Data in the basic school mathematics curriculum.
- reflect on how Geometry and Handling Data were taught in their basic school days and compare with current practice in basic schools.
- reflect on the core skills and competencies (e.g. communication and collaboration, critical thinking and problem solving, digital literacy) teachers need to develop to make them good teachers.
- engage in peer assessment on awareness of core skills and competencies needed to enhance own strengths and address limitations regarding the teaching and learning of Geometry and Handling Data.

**Weighting: 40%**

**Assesses Learning Outcome(s):** CLO 1- 4 (NTS 2b, 3j)

**Component 3: Coursework 2**

**Summary of Assessment Method:**

Self-Assessment (as part of their portfolio): Student-teachers should be given an assessment tool or questionnaire at the onset and the end of the course to

- do self-assessment and compare their attitude towards learners, mathematics teaching and readiness to support learners who have misconceptions or struggle with the subject.
• do self-assessment and compare their value as well as respect for equity and inclusivity in the mathematics classroom.
• reflect critically on their own learning experiences and use them to plan for their own continuous personal development.
• identify and reflect on mathematics related learning difficulties within the context of Geometry and Handling Data.

Weighting: 20%
Assesses Learning Outcome(s): CLO 3, 4 (NTS 1a, 2f)

TEACHING/ LEARNING RESOURCES

<table>
<thead>
<tr>
<th>Teaching/ Learning Resources</th>
<th>Maths posters</th>
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<tbody>
<tr>
<td></td>
<td>Manipulatives and visual aids</td>
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<tr>
<td></td>
<td>Computers and other technological tools</td>
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<td></td>
<td>Set of Mathematical instruments</td>
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<td></td>
<td>Geoboard (Geodot)</td>
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</tbody>
</table>


UNIT ONE

PLANE GEOMETRY (PATTERNS IN SHAPE): LEARNING, TEACHING AND APPLYING

Introduction

You are welcome to the first unit of this course in Learning, Teaching and Applying Geometry. This unit looks at fundamental concepts in Geometry of which angles, triangles, quadrilaterals, polygons and tessellations are introduced.

You will agree with me that in a fundamental sense, geometry is a natural outgrowth of our exposure to the physical universe and in particular to the natural world. In your interaction with your environment, you encounter physical shapes, such as buildings and other objects. These you can organize by Patterns into groups and classes. You can put the sun and moon into ‘‘round’’ category and building into another separate category. The concepts of point, lines, planes, rays, lines and line segment play an important role in geometry. They provide clarity, and consistency in categorizing geometrical shapes. Have you read about them? Do not get worried if you have not done so. This unit will introduce you to these concepts. It is our hope that you will have no difficulties in understanding the contents of the unit.

In this Unit, we will look at Learning, Teaching and Applying Geometry made up of the following Sections:

Section 1.1 Angles at a point and angles and parallel lines
Section 1.2 Angles and triangles and their properties
Section 1.3 Properties of quadrilaterals and other polygons
Section 1.4 Learning about 3-Dimensional shapes: comparing polyhedral, forming 3-dimensional shapes
Section 1.5 Learning about 2-dimensional shapes.
Section 1.6 Tessellations and applying these to the teaching of the JHS Mathematics curriculum

It is my hope that after studying this unit you will:

- Demonstrate knowledge and understanding of fundamental concepts in plane geometry as found in the basic school mathematics curriculum.
• Use manipulatives and other TLMs and models in a variety of ways in developing and learning geometrical concepts.
• Develop an appreciation for geometry as a means of describing the physical world
• Demonstrate awareness of the world outside the classroom as a rich source of geometrical ideas

I think this unit will equip you with enough knowledge that can help you improve your teaching of geometry in school mathematics.

Let us move on to section 1 of this unit.
SECTION ONE: ANGLES AT A POINT AND ANGLES AND PARALLEL LINES

Introduction: This section aims to present to you Angles at a point and angles and parallel lines

Objectives: After completing this section 1 you will

- Interpret and analyse definitions of given plane geometry concepts that appear in the basic school mathematics curriculum.
- Identify, select (or design) and use manipulatives and other TLMs and models to develop geometrical concepts, as well as, suggest such materials can be used to teach given concepts in the basic school curriculum
- Outline the usefulness of geometry and justify why it should be taught both orally and in writing

What fundamental concepts related to points, lines and angles and their properties do you know?

What riddles do you know about points, lines, line segments, rays, and angles?

Why do you think plane geometry should be studied in school?

Definitions and interpretations of geometric concepts.

Hello my dear students. I believe you have some knowledge about what a point is but this knowledge may be informal. In this section we will begin to know more about a point in geometry. Today a formal presentation of Euclidean geometry begins with a description of points, lines, and planes. ‘’point’’, ‘’line’’, and ‘’plane’’ are undefined terms that we can understand using everyday objects and our intuition. Points, lines, and planes are suggested by our surroundings.

A point is a location in space. It has neither height nor length. Then what is a line: two points determine a line, which is a set of points that follow a path going indefinitely in both directions. A portion of a line that has two endpoints is a line segment. So then what is a ray: A ray is a section of a line that has only one endpoint. A line is identified by any two points on the line, but is can also be given a name using a single lowercase letter. Points are identified as uppercase letters. A line segment is identified by its two endpoints. A ray is identified by the endpoint, which is written
first, and another point on a ray.

**Plane:** a plane is a collection of points lying on a flat surface, which extends indefinitely in all directions.

Parallel lines are equidistant (the same distance) from each other and never intersect. Right angles are formed when two lines intersect and are perpendicular to each other. The symbol for a right angle is a small square at the vertex of the angle (the endpoint shared by the two segments that form the angle).

When segments, rays or lines meet or intersect, angles are created. Angles whose measure is between $0^\circ$ and $90^\circ$ are called **acute angles**. A $90^\circ$ angle is called a **right angle**. If the measure of an angle is between $90^\circ$ and $180^\circ$, the angle is an **obtuse angle**. An angle whose measure is exactly $180^\circ$ is called a **straight angle**.
A protractor is used to measure angles. There are two rows of numbers on a protractor. Place the crosshairs of the protractor on the vertex of the angle so that the horizontal line of the crosshairs lies on one of the rays of the angle. The other ray of the angle will fall on one of the angles. The other ray of the angle will fall on one of the numbers of the protractor. If the ray lands on the numbers 70° and 110°, the measure of the angles is 70° if the angle is acute or 110° if it is obtuse. The angle in the next diagram Measures 110° because it is an obtuse angle.

Use your protractor to verify these measurements. The rays that form angles continue indefinitely. Extend the rays if you need more length to measure an angle. If the sum of two angles is 90°, the angles are complementary angles. Think of complementary and corner. The angles form a right angle like the inside corner of a room. The complement of a 67° angle is a 23° angle (67° + 23° = 90°). The complement of a 12° angle is a 78° angle (12° + 78° = 90°). If the sum of two angles is 180°, the angle are supplementary angles. Think of supplementary and straight. The angles form a straight angle that appears as a straight line. The supplementary of a 146° angle is a 34° angle (146° + 34° = 180°). The supplement of a 89° angle is a 91° angle (89° + 91° = 180°). Two intersecting lines form four angles. Vertical angles are equal in measure. Angles that have the same measure are congruent. In the diagram below, angle a and c are vertical angles, as are angles b and d. Any two of these angles are either congruent or supplementary. Suppose the measure of angle a is 40°. That makes angle b its supplement with a measure of 140°. Angles b and c are supplementary. If angle b measures 140°, then angle c measures 40°. You can continue this discussion for angles c and d and for angles d and a. In this case, the pair of vertical angles a and c are 40°, and angles b and d are each 140°.
When two parallel lines are cut by a *transversal* (a line that cuts across parallel lines), eight (separate) angles are formed. Any two of these angles are either supplementary or congruent.

Imagine that you can place line $m$ on $k$. Then the pairs of angles 1 and 5, 2 and 6, 3 and 7, and 4 and 8 are called corresponding angles and are congruent. Angles 3 and 6, and angles 4 and 5, are called *alternate interior angles*. They are on opposite sides of the transversal and are between the two parallel lines. These are congruent. Angles 2 and 7, and angles 1 and 8, are called *alternate exterior angles*. And they also are congruent. These angles are on opposite sides of the transversal but lie outside the parallel lines. If the measure of angle 3 is $30^\circ$, you can find the measure of angle 8. You know that angles 3 and 7 are corresponding angles and that angles 7 and 8 are supplementary. Thus angle 8 measures $150^\circ$ *that is* $(180^\circ - 30^\circ = 150^\circ)$.

**Activities**

1. Find the angles marked a, b, and c in the diagram below.
2. Find the sizes of the angles marked a, b, c and d in the diagram below.

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<tbody>
<tr>
<td>a</td>
<td>b</td>
<td>c</td>
<td>d</td>
</tr>
<tr>
<td>70°</td>
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Summary

Geometry is built upon a series of undefined terms. These terms are those which we accept as known in order to define other terms.

**Point:** Although we represent points on paper with small dots, a point has no size, thickness, or width.

**Line:** A line is a series of adjacent points which extends indefinitely. A line can be either curved or straight; however unless otherwise stated, the term “line” refers to a straight line.

**Plane:** A plane is a collection of points lying on flat surface, which extends indefinitely in all directions.
The ceiling and floor of a room suggest parallel planes while the ceiling and side wall of a room suggest perpendicular plane.

**Ray:** A ray is a series of points that lie to one side of a single endpoint.

**Angle:** An angle is a degree measure of a turn.

**Vertical angles:** Vertical angles are formed when two lines intersect. These angles are equal

**Adjacent angles:** Adjacent angles are two angles with a common vertex and a common side, but no common interior points.

**Right angle:** A right angle is an angle whose measure is 90°

An **acute angle:** An acute angle is an angle whose measure is larger than 0°, but less than 90°

An **obtuse angle:** An obtuse angle is an angle whose measure is larger than 90° but less than 180°

A **straight angle:** A straight angle is an angle whose measure is 180°. Such an angle is, in fact, a straight line.

A **reflex angle:** A reflex angle is an angle whose measure is greater than 180° but less than 360°.

**Complimentary angles:** Complimentary angles are two angles whose measures total 90°

**Supplementary angle:** Supplementary angle are two whose measures total 180°

**Congruent angles:** Congruent angles of equal measure.

Classifying angles according to their relative measures through exploration and mathematical discourse.
SECTION TWO: ANGLES AND TRIANGLES AND THEIR PROPERTIES

Introduction

Dear learners, you are warmly welcome to unit 1 of section 2. In this section, we would discuss angles and triangles and their properties. We advise that you investigate these properties with your pupils in class as a prospective teachers.

Indicators: after going through this section you will be able to:

- Interpret and analyse types and properties of angles and triangles of given plane geometry concepts that appear in the basic school mathematics curriculum.
- Identify, select (or design) and use manipulatives and other TLMs and models to develop geometrical concepts, based on angles, triangles and their properties.
- Outline the usefulness of geometry and justify why it should be taught both orally and in writing.

Exploring the types and properties of triangles

The construction approach

Exploring different ways of classifying triangles and their properties. (Paper folding, designs in fabrics, etc)

You can explore using other cultural artefacts.

TRIANGLES

Suppose we are given the lengths of three lines segments. Can we build a triangle with these segments? This introduces us to an interesting property of triangles which states that the sum of the lengths of any two sides is always greater than the length of the third side. You can have a triangle whose sides are 4, 5, and 7 because (4+5) > 7, (4+7) > 5, and (5+ 7) > 4. You cannot have a triangle whose sides are 3, 4, and 10 because (3+ 4) < 10. Take thin strips of paper that are 3 inches, 4 inches, and 10 inches long. Then try to arrange the strips to create a triangle. It can’t be done!
Tell whether it is possible to form a triangle with sides of the given lengths.

1. 18, 14, 28
2. 6, 80, 80
3. 33, 19, 14
4. $12\frac{1}{2}$, $28\frac{1}{2}$, $15\frac{3}{4}$

All triangles have three angles and three sides. An **equilateral triangle** has three congruent sides 
and three angles equal in measure. An **isosceles triangle** has two congruent sides and two congruent 
angles. A scalene triangle has three sides of different lengths. One way to classify triangles is by 
how many congruent sides they have. **Scalene triangles** have no congruent sides; **isosceles 
triangles** have at least two congruent sides; and **equilateral triangles** have three congruent sides.

A second way to classify triangles is by their largest angle. All triangles have at least two acute 
angles. A **right triangle** has one right angle; an **obtuse triangle** has one obtuse angle; and all the 
angles are acute in an **acute triangle**. By combining the terms **acute**, **obtuse**, and **right** with 
equilateral, isosceles, and scalene, we can describe any triangle. Therefore we can classify 
triangles according to both sides and angles. Sketch an example of each of the following, or write 
‘‘impossible’’ if appropriate.

a) Scalene triangle  
b) Isosceles scalene triangle

c) Isosceles obtuse triangle  
d) Isosceles right triangle

e) Obtuse right triangle  
f) Equilateral acute triangle

**ACTIVITIES**

1. Explain why it is impossible to have a triangle whose three sides are 6cm, 8cm, and 15cm.

2. Find the measure of the third angle of the triangle is the first two angles contain
   
a) 60°, 40°
   
b) 100°, 20°
   
c) 50°, 70°

3. Show whether the three angles can be the three angles of a triangle.
   
a) 30°, 70°, 80°
   
b) 70°, 80°, 90°
   
c) 30°, 110°, 40°

4. Find the number of degrees in each angle of an equiangular triangle.
5. Is it possible to have: a) two right angles? b) two obtuse angles? c) one right and one obtuse angle? Why and why not?
6. What is the sum of the measures of the two acute angles of a right triangle?
7. If two angles in one triangle contain the same number of degrees as two angles in another triangle, what must be true of the third pair of angles in the two triangles? Why?
8. In each part find the measures of the three angles of a triangle whose measures are in the given continued ratio.
   a) 1: 2 : 3  b) 1: 4 : 7  c) 2 : 3 : 4  d) 4 : 5 : 9  e) 1 : 3 : 4
9. In a triangle the measure of the second angle is 3 times the measure of the first angle, and the measure of the third angles is 5 times the measure of the first angle. Find the number of degrees in each angles of the triangle.
10. The longest side of a triangle is opposite the angles; the shortest side is opposite the smallest angle.
11. If one angle of a triangle is obtuse, can another also be obtuse? Why or why not?
12. If one angle in a triangle is acute, can the other two angles also be acute? Why or why not?
13. Can a triangle have two right angles? Why or why not?
14. If a triangle has one acute angle, is the triangle necessarily acute? Why or why not?

Summary

In this section, we look at angles and triangles and their properties

- Triangles are closed figures containing three angles and three sides area is the measure of the total surface of an object.
- When two straight lines meet at a point, they form an angle.
- Angles in any triangle add to 180°

SECTION THREE: PROPERTIES OF QUADRILATERALS AND OTHER POLYGONS.
Introductions
Hello our hardworking learner, you welcome to section unit 1 of section three. In the previous section, you learnt about angles and triangles and their properties. We hope you have revised section two and other previous sections extensively. In this section, you will learn about the idea of properties of quadrilateral and other polygons.

INDICATORS: After going through this section you will be able to:

- Interpret and analyse definitions and properties of quadrilaterals and other polygons that appear in the basic school mathematics curriculum.
- Identify, select (or design) and use manipulatives and other TLMs and models to develop properties of quadrilaterals and other polygons, as well as, suggest such materials can be used to teach given concepts in the basic school curriculum.
- Outline the usefulness of quadrilaterals and other polygons and justify why it should be taught both orally and in writing.

Exploring types of quadrilaterals and their properties.
Properties of special quadrilaterals

QUADRILATERALS
I am sure you know that a quadrilateral is a four-sided plane figure. What about how they are classified? Quadrilaterals are classified by their attributes. A point at which any two sides of the quadrilateral meet is called a vertex of the quadrilateral. At each vertex the two sides that meet form an angle of the quadrilateral. Let us look at some examples of quadrilaterals.

Parallelograms
If a quadrilateral has two pairs of opposite sides that are parallel, then the quadrilateral is a parallelogram. A rectangle has two pairs of parallel sides, four right angles, and congruent diagonals. A square is a special instance of a rectangle that has four congruent sides, four congruent right angles, and congruent diagonals. A rhombus is a parallelogram with four congruent sides.

Trapezoid
If a quadrilateral has one pair of parallel sides, it is a trapezoid. Depending on how it is constructed,
a trapezoid may or may not have a right angle. It will be an isosceles trapezoid where the nonparallel sides are equal in length and the base angles are equal.

A rectangle is parallelogram in which all four angles are right angles. A rhombus is also a parallelogram in which all the sides are equal in length. A square is a rectangle all of whose sides are equal in length. We can also describe a square as a rectangle which is a rhombus. It is also possible to describe a square as a rhombus which has four right angles.

POLYGONS
Poly- is Greek words for ‘’many’’ and –gon comes from the Greek word for knee, bend, or angle. Polygons get their names from the number of angles they have. Since they have the same number of angles as sides, we identify a polygon by the number of sides it has.

<table>
<thead>
<tr>
<th>Numbers of Angles</th>
<th>Names of Figure</th>
</tr>
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<tbody>
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<td>3</td>
<td>Triangle</td>
</tr>
<tr>
<td>4</td>
<td>Quadrilateral</td>
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<tr>
<td>5</td>
<td>Pentagon</td>
</tr>
<tr>
<td>6</td>
<td>Hexagon</td>
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<tr>
<td>7</td>
<td>Heptagon</td>
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<tr>
<td>8</td>
<td>Octagon</td>
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<tr>
<td>9</td>
<td>Nonagon</td>
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<tr>
<td>10</td>
<td>Decagon</td>
</tr>
<tr>
<td>11</td>
<td>Undecagon</td>
</tr>
<tr>
<td>12</td>
<td>Duo-decagon</td>
</tr>
</tbody>
</table>
Similarity and Congruence

Two figures are congruent if they have the same size and shape. Congruent line segments have the same length. Congruent angles have the same measure in degrees. All right angles are congruent because every right angle measures $90^\circ$. The symbol $\cong$ means congruent. Two triangles are similar if their corresponding angles are congruent and the lengths of corresponding sides have the same ratio. Triangle $ABC$ has sides with length with length of 3, 4, and 5 units. Triangle $A'B'C'$ has sides with length of 6, 8, and 10 units.

Activities

State whether the following statement is true or false?

1. If a polygon is a trapezoid, it is a quadrilateral.
2. If a polygon is a rectangle, it is a parallelogram
3. If a polygon is a rhombus, it is a parallelogram
4. If a polygon is a parallelogram, it is a square.
5. If a polygon is a rhombus, it is a square.
6. If two angles are opposite angles of a parallelogram, they are congruent.

SUMMARY

1. Rectangles, rhombuses, and squares are members of the family of parallelograms. Therefore, any property of the family of parallelograms must also be a property of rectangles, rhombuses, and squares.
2. A square is a member of the family of rectangles. Therefore, any property of the family of rectangles must also be a property of squares.
3. A square is a member of the family of rhombuses. Therefore, any property of the family of rhombuses must also be a property of square.
4. All squares are parallelograms, but not all parallelograms are squares.
5. All squares are rectangles, but not all rectangles are squares.
6. Every square is a rhombus, but not all rhombuses are squares.
7. Opposite sides of a parallelogram are equal in length (or congruent).
8. Opposite angles of a parallelogram are equal in measure (or congruent).
9. Consecutive angles of a parallelogram are supplementary.
SECTION FOUR: LEARNING ABOUT 3-DIMENSIONAL SHAPES: COMPARING POLYHEDRAL, FORMING 3-DIMENSIONAL SHAPES.

Introduction

You are welcome to section four of the first unit of this course. How did you find the previous section? I hope you found it quite interesting. Section four promises to be even more interesting. This is because most of the things discussed in this section are familiar to you. This section in a way revises geometrical shapes. Geometrical shapes make up everything around us. Geometrical shapes can be both two-dimensional, like the screen of your computer, and three-dimensional, like a child’s ball. Each geometric shape has its own properties that make it different from other shapes. However geometric shapes may share properties with other shapes, requiring them to be further described to distinguish them from other shapes.

INDICATORS: After going through this section you will be able to:

- Interpret and analyse definitions of 3-Dimensional shapes that appear in the basic school mathematics curriculum.
- Identify, select (or design) and use manipulatives and other TLMs and models to develop 3-Dimensional shapes, as well as, suggest such materials that can be used to teach given concepts in the basic school curriculum

3-DIMENSIONAL SHAPES

Recall that polygons and polygonal regions are different. A polygon refers to the boundary, whereas a polygonal region is the union of the boundary and the interior. Similarly, a distinction is made with three-dimensional figures. Most of the geometric figures you have worked with so far have been flat plane figures with two dimensions—base and height. In this sections you will work with solids, like rocks and plants, are very irregular, but many others are geometric. Some real-world geometric solids occur in nature: viruses, oranges, crystals, the earth itself. Others are human-made: books, buildings, footballs, milk tins, ice cream cones. Three-dimensional geometry plays an important role in the structure of molecules. For example, when carbon atoms are arranged in a very rigid network, they form diamonds. A solids formed by polygons that enclose a single region of space is called a polyhedron. The flat polygonal surfaces of a polyhedron are called its faces. Although a face of a polyhedron
includes the polygon and its interior region, we identify the face by naming the polygon that encloses it. A segment where two faces intersect is called an edge. The point of intersection of three or more edges is called a vertex of the polyhedron. Just as a polygon is classified by its number of sides, a polyhedron is classified by its number of faces. The prefixes for polyhedron are the same as they are for polygons with one exception: A polyhedron with four face faces is called a tetrahedron. Here are some examples of polyhedrons: Hexahedrons, Heptahedrons and Decahedrons. If each face of a polyhedron is enclosed by a regular polygons, and each face is congruent to the other faces, and the faces meet at each vertex in exactly the same way, then the polyhedron is called a regular polyhedron.

**Prism**
A prism is a polyhedron formed by two congruent polygonal bases in parallel planes connected by three or more parallelogram-shaped regions. A prism with lateral faces that are rectangular regions is a right prism.

**Pyramid**
A pyramid is a polyhedron that has a polygonal base. Its lateral faces are triangular regions with a common vertex.

The following table summarizes the different polyhedra and the numbers of faces, vertices, and edges they have.

<table>
<thead>
<tr>
<th>Figure</th>
<th>Faces</th>
<th>Vertices</th>
<th>Edges</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cube</td>
<td>6</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>Triangular prism</td>
<td>5</td>
<td>6</td>
<td>9</td>
</tr>
<tr>
<td>Rectangular prism</td>
<td>6</td>
<td>8</td>
<td>12</td>
</tr>
<tr>
<td>Pentagonal prism</td>
<td>7</td>
<td>10</td>
<td>15</td>
</tr>
<tr>
<td>Triangular pyramid</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Square pyramid</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Models of 3-D shapes and their nets.

Cylinder

Net of a cylinder

Pyramid

Net of a pyramid
Cone

Activities

1. Prisms and pyramids are special types of polyhedra. In what ways are polyhedral like polygons? In what ways are they different?

2. What is the least number of faces that can meet at a vertex of any polyhedron? What is the least number of edges that can meet at a vertex of any polyhedron?

3. How are pyramids like prisms? How are they different?

4. Sketch a net for a pyramid in which the base is five-sided. Name the pyramid.

5. As the number of sides in the base of prism increases, what shape does the prism approach?

6. As the number of sides in the base of a pyramid increases, what shape does the pyramid approach?

7. Complete the table

<table>
<thead>
<tr>
<th>Regular polyhedron</th>
<th>Faces</th>
<th>Vertices</th>
<th>Edges</th>
</tr>
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<tbody>
<tr>
<td>Cube</td>
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<tr>
<td>Tetrahedron</td>
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<tr>
<td>Octahedron</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>Icosahedron</td>
<td>12</td>
<td>30</td>
<td></td>
</tr>
</tbody>
</table>
SUMMARY
In this section, we have name and classified the basic kinds of three-dimensional geometric figures. From the explorations, you can see that a variety of seemingly different figures are actually prisms. You have seen that prisms and cylinders are closed related. You have seen that pyramids and cones are closely related.
SECTION FIVE: LEARNING ABOUT 2-DIMENSIONAL SHAPES: POLYGONS \((n \geq 3)\)

Introduction

This the fifth section of the first unit. You are going to look at two-dimensional shapes. Since you have already looked at three-dimensional shapes in section 4, this section should not be difficult.

**LEARNING INDICATORS:** After going through this section you will be able to:

- Interpret and analyse definitions of 2-Dimensional shapes that appear in the basic school mathematics curriculum.
- Identify, select (or design) and use manipulatives and other TLMs and models to develop 2-Dimensional shapes, as well as, suggest such materials can be used to teach given concepts in the basic school curriculum
- Outline the usefulness of geometry and justify why it should be taught both orally and in writing
- Present and analyse designs in our fabrics and pictures of architectural and artistic, designs that can be found in our communities
- Outline the influence of translation, rotation, and reflection on polygons in a plane;
- Recognise and analyse the importance of symmetry, congruence and similarity in real life situations

**TWO-DIMENSIONAL FIGURES**

Think of geometric figures that people generally find pleasing. In this section, we will learn more about geometric figures and shapes and you will learn and discover some of the secrets of the patterns in the objects you have seen. We will examine how shapes can be put together to make interesting designs you have seen, such as those in quilts, plants and animals, and buildings. Let us look at some examples of two-dimensional figures.

**SQUARES**

Each figure below is made up of four squares.

![Squares Diagram]
A square is a quadrilateral with some special characteristics.

- First, all of sides are congruent. This is shown by the matching black on the sides of the figure.
- Second, a square has four right angles. This is shown by the red right symbols symbols
- Finally, pairs of opposite sides are parallel. You can extend the sides, and the opposite sides will never intersect.

**PROPERTIES OF A SQUARE**

- Opposite sides are parallel
- All sides are equal in length
- All angles are equal
- All angles are right angles
- Adjacent sides are perpendicular
- Diagonal bisect each other
- Diagonal are equal in length
- Diagonals bisect each other at right angles
- Diagonals bisect the angle

**PROPERTIES OF RECTANGLE**

- Opposite sides are parallel
- Opposite angles are equal (or congruent)
- Opposite sides are equal in length
- All four angles are right angles
PROPERTIES OF RHOMBUS

- All sides are equal in length
- Opposite sides are parallel
- Opposite angles are equal (or congruent)
- Diagonals bisect each other
- Diagonals bisect angles
- All angles are not right angles

Activities

1. In what ways are a square and rectangle alike? In what ways are they different?
2. In what ways are a square and parallelogram alike? In what ways are different?
3. In what ways are the triangle and hexagon alike? In what ways are different?

Summary

In this section we learnt about
• The nature of side lengths can determine the types of a geometrical shape.
• Properties of some triangles
• Properties of some special quadrilaterals
SECTION SIX: TESSELLATIONS AND APPLYING THESE TO THE TEACHING OF THE JHS MATHEMATICS CURRICULUM.

Introduction
What do you know about polygons that tessellate and those that cannot tessellate a plane? Looking around your house, school or other places what designs have you come across that portray tessellation? In this section we shall learn about tessellations and how to apply tessellations to the teaching of the JHS mathematics curriculum.

LEARNING INDICATORS: After going through this section you will be able to:
- Outline the usefulness of geometry and justify why it should be taught in school
- Identify, select (or design and use manipulatives and other TLMs and models to develop geometrical concepts.

TESSELLATIONS AND POLYGONS
Honeycombs are remarkably geometric structures. Hexagonal cells that bees make are ideal because they fit together perfectly without any gaps. The regular hexagon is one of many shapes that can completely cover a plane without gaps or overlaps. Mathematicians call such an arrangement of shape a tessellation or a tiling. A tessellation that uses only one shape is called a monohedral tiling. You can find tessellations in every home. Decorative floor tiles have tessellating patterns of squares. Brick walls, fireplaces, and wooden decks often display creative tessellations of rectangles. Where do you see tessellations every day? You already know that squares and regular hexagons create monohedral tessellations. Because each regular hexagon can be divided into six equilateral triangles, we can logically conclude that equilateral triangles also create monohedral tessellations. Will other regular polygons tessellate? Let’s look at this questions logically. For shape to fill the plane without gaps or overlaps, their angles, when arranged around a point, must have measures that add up to exactly 360°. If the sum is less than 360°, there will be gap. If the sum is greater, the shapes will overlap. Six 60° angles from six equilateral triangles add up to 360°, just as four 90° angles from four squares or three 120° angles from three regular hexagons. What about regular pentagons? Each angle in a regular pentagon measure 108°, and 360° is not divisible by 108. So regular pentagons cannot be arranged around a point without
overlapping or leaving a gap. What about regular heptagons? In any regular polygon with more than six sides, each angle has a measure greater than 120°, so no more than two angles can fit about a point without overlapping. So the only regular polygons that create monohedral tessellations are equilateral triangles, squares, and regular hexagons. A monohedral tessellation of congruent regular polygons is called a regular tessellation. Tessellations can have more than one type of shape. You may have seen the octagon-square combination at right. In this tessellation, two regular octagons and a square meet at each vertex. Notice that you put your pencil on any vertex and the point is surrounded by one square and two octagons. So you can call this a 4.8.8 or a 4.8² tiling. The numbers give the vertex arrangement, or numerical name for tiling. When the same combination of regular polygons (two or more kinds) meet in the same order at each vertex of a tessellation, it is called a semi-regular tessellation.

Activity
1. What do we mean when we say a shape tessellate?

2. Briefly explain the term regular tessellation

Summary
We have come to the end of section 6 and Unit 1. What have you learnt in this section? Remember that a tessellation that uses only one shape is called a monohedral tiling. A monohedral tessellation of congruent regular polygons is called a regular tessellation. Tessellations can have more than one type of shape for a shape to fill the plane without gaps or overlaps, their angles, when arranged
around a point, must have measures that add up to exactly 360°. I am sure you are conversant with the concept of tessellation. Well done!
UNIT TWO
GEOMETRICAL CONSTRUCTIONS

Learning, teaching and applying

SECTION ONE: CONGRUENCE, SIMILARITIES AND CONSTRUCTIONS
(TEACHING SYMMETRY, CONGRUENCE, SIMILARITY AND CONSTRUCTION OF SHAPES).

Welcome to the first section of Unit 2. The whole unit will focus on teaching geometrical constructions. In this section, we are going to have a look at congruence, similarities and constructions (teaching symmetry, congruence, similarity and construction of shapes). Before we proceed, let us ask ourselves: what is congruence and similarity of shapes? I hope you will enjoy this section to the fullest.

LEARNING OUTCOMES

• Recognize professions and artisans who use the knowledge and skills derived from geometric construction in their work
• Demonstrate knowledge and understanding of fundamental concepts based on constructions of angles as found in the basic school mathematics curriculum.
• Demonstrate awareness of the world outside the classroom as a rich source of geometrical ideas

LEARNING INDICATORS

• Examine the usefulness of geometric construction and be able to justify the need for studying this in basic school.
• Identify and provide sound argument about angles that are possible to construct and those that cannot be constructed.
• Identify, select (or design) and use appropriate tools and equipment that can be used in solving problems based on geometric construction and loci in the basic school mathematics curriculum.

Symmetry, congruence, similarity and construction of shapes

The concepts of congruence, similarity, and symmetry can be understood from the perspective of geometric transformation. Fundamental are the rigid motions: translations, rotations, reflections, and combinations of these, all of which are here assumed to preserve distance and angles (and therefore shape in general). Reflections and rotations each explain a particular type of symmetry,
and the symmetries of an object offer insight into its attributes as when the reflective symmetry of an isosceles triangle assures that its base angles are congruent.

**Symmetry**

In geometry, a figure is symmetrical if an operation can be done to it that leaves the figure occupying an identical physical space. This can be accomplished in two ways (line of symmetry and rotational symmetry). Line symmetry occurs when a line may be passed through an object such that both halves of the object perfectly mirror each other. Consider the triangle below. An equilateral triangle has three instances of line symmetry: one from each vertex to the midpoint of the opposite side.

Rotational symmetry occurs when a shape may be rotated to occupy the same space as the original. Let’s take the same triangle again. An equilateral triangle has three degrees of rotational symmetry: at 120, 240, and 360 degrees. Once we have determined rotational symmetry to 360 degrees, we can stop, as the pattern will repeat itself after that.

Triskelions is also an example of rotational symmetry.
Introducing Similarity

A similarity transformation is a transformation in which the image has the same shape as the pre-image. Specifically, the similarity transformation are the rigid motions (reflection, translations and rotation) as well as dilations.

Two plane figures are similar if and only if one can be obtained from the other by similarity transformations (that is by a sequence of reflection, translation, rotation and dilations). The symbol of similarity is ~. As with congruence, it is customary to write a similarity statement so that corresponding vertices of the figures are listed in the same order.

If ΔA¹B¹C¹ is the image of ΔABC after a dilation with centre O and scale factor k, then the two triangles are similar and you write ΔABC ~ΔA¹B¹C¹.

Congruence figures or shapes

Two geometric figures are defined to be congruent if there is a sequence of rigid motions that carries one onto the other. This is the principle of superposition. For triangles, congruence means the equality of all corresponding pairs of sides and all corresponding pairs of angles. During the middle grades, through experiences drawing triangles from given conditions, students notice ways to specify enough measures in a triangle to ensure that all triangles drawn with those measures are
congruent. Once these triangle congruence criteria are established using rigid motions, they can be used to prove theorems about triangles, quadrilaterals, and other geometric figures.

**Applying Similarity and Congruence to Triangles.**

Recall that when two figures are similar, there is a sequence of similarity transformation that maps one figure to the other. In particular, gives ΔABC \( \sim \) ΔDEF, you can first a dilation to ΔABC to make both triangle the same size. Then you can apply a sequence of rigid motions to the dilated image of ΔABC to map it to DEF.

Because the similarity transformations that map ΔABC \( \sim \) ΔDEF preserve angle measure, you can say that corresponding angles are congruent. Thus, ΔABC \( \sim \) ΔDEF implies,

Also, the initial dilation that makes the two triangles the same size shows that each side of ΔDEF is larger or shorter than the corresponding side of ΔABC by the ratio given by the scale factor. Assuming the dilation has the scale factor \( k \), this means that:

\[
DE = k \cdot AB, \quad EF = k \cdot BC, \quad \text{and} \quad DF = k \cdot AC
\]

Solving for \( k \), in these equations gives

\[
k = \frac{DE}{AB}, \quad k = \frac{EF}{BC} \quad \text{and} \quad k = \frac{DF}{AC}
\]

This shows that corresponding sides are proportional. That is:
Example: Identifying congruent angles and proportional sides.

Given that \( \Delta RST \sim \Delta UVW \), write congruence statements for the corresponding angles and proportions for the corresponding sides.

A. Corresponding angles are listed in the same position in each triangle name.

\[ \angle R \equiv \angle U, \quad \ldots \ldots \]

B. Corresponding sides are pair of letters in the same position in each triangle name.

\[ \Delta RST \sim \Delta UVW \]

You have seen that when two triangles are similar, corresponding angles are congruent and corresponding sides are proportional. The converse is also true. That is if you are given two angles and you know that the corresponding angles are congruent and corresponding sides are proportional, you can conclude that the triangles are similar.

As with congruence, there are some shortcuts that make it a bit easier to prove that two triangles are congruent. The most important of these is known as the AA similarity.

**AA Similarity Criterion**

If two angles of one triangle are congruent to two angles of another triangle, then the triangles are similar.

Given: \( \angle A \equiv \angle X \) and \( \angle B \equiv \angle Y \)

Prove: \( \Delta ABC \sim \Delta XYZ \)
Activities

1. Explain why congruence can be considered a special case of similarity.

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2. If you know that two figures are similar, can you conclude that corresponding angles are congruent? Why or why not?

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3. A student identified $\overline{RS}$ and $\overline{UV}$ as corresponding sides. The student wrote $\frac{RS}{UV} = \frac{VW}{ST}$. Is this a correct proportion? Why or why not? If the proportion is not correct, explain how to write correctly.

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4. Is triangle similarity transitive? That is, if $\triangle ABC \sim \triangle DEF$ and $\triangle DEF \sim \triangle GHK$, can you conclude that $\triangle ABC \sim \triangle GHK$? Explain.

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5. What can you conclude about similar triangles and how can you prove triangles is similar?

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Summary
We have come to the end of section 1 of Unit 2. What have you learnt in this section? Remember that similarity transformation is a transformation in which the image has the same shape as the pre-image. Two plane figures are similar if and only if one can be obtained from the other by similarity transformations (that is by a sequence of reflection, translation, rotation and dilations). I am sure you are conversant with symmetry, similarity and congruence of shapes in geometry. Congratulations!
SECTION TWO: TEACHING MEASUREMENT OF A LINE

Hello my dear students. It is very assuring that we are progressing steadily through this course. How have you been doing so far with your study and the activities. Well done! I wish to remind you that geometric construction is an important concept in geometry. In the previous section, we looked at the concept of congruence and similarity in geometry. Am sure by now you are conversant with these concepts. In this section, we are going to discuss construction and bisecting a line segment, and constructing a perpendicular from a point outside a line to the line. Let’s look at these concepts in details.

Geometric Construction

An important part of Euclidean geometry is the construction of plane geometric figures using only two tools. Straightedge and a compass. A straightedge is similar to a ruler except that it does not contain any marks used for measuring length. A compass is a tool that can be used to copy a line segment, draw a circle of a given radius, and draw an arc. We will now demonstrate the use of both of these tools, through seven simple constructions.

Construction of a line

Let us look at how to construct a line segment.

Let us look at how to construct line AB, with $|AB| = 5$cm.

![Line AB with 5cm length](image)

**Procedure**

a. Using a sharp pencil and a ruler, draw a line of length longer than 5cm
b. With the aid of a ruler and a pencil mark the line towards the left end and label that point A
   c. With your pair of compasses, measure 5cm on your ruler
d. With the point of compasses at A, draw an arc to cut the line towards the right hand end.
e. Label the intersection of the arc and the line as B.
f. Using a pair of dividers, check that the line segment AB constructed is of length 5cm.

Bisecting a given line segment

Let us look at how to construct a given line segment.

![Diagram of line segment AB with points C, M, and D marked.]

Procedure

a. Construct the line segment AB of a given length
b. Open your pair of compasses to a radius greater than half the length of AB.
c. Place the point of the compass at A, draw two arcs, one above the line and one below the line.
d. With exactly the same radius, place the point of the compass at B, draw two more arcs to intersect the first two arcs drawn already. The two points of intersection are labelled as C and D.
e. Join C to D with a straight line.
f. Label the intersection of CD and AB as M. M is the midpoint of AB and at that point CD and AB are at right angles. CD is therefore, called a perpendicular bisector of AB and also known as the mediator.
Constructing a perpendicular from a point outside a line to the line.

Procedure

a. Draw a line AB
b. Choose a point P above AB
c. Open your pair of compasses to a convenient radius and place the point a P, construct two arcs to cut AB at C and D.
d. With centre C and a convenient radius, draw an arc to cut the first one at Q.
e. Join P and Q to cut AB at E
f. PE is the perpendicular from P to AB and the shortest distance from P to AB.

Activities

1. Briefly explain how to construct the following:
   a. A line segment
   b. A perpendicular from a point outside a line to the line.
2. How will you bisect a given line segment?

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Summary

Let us pause here and reflect on what we have learnt so far. What have we covered in this section? In this section, we dealt with how to construct and bisect a given line segment and how to construct a perpendicular from a point outside a line to the line. I am sure you have enjoyed our discussion. Let us now look at the third section of Unit 1.
Hello my dear students, well done for working successfully through the first two sections. I welcome you to section 3 of Unit 2. In this section, we will focus on another important aspect of geometric construction that deals with bisecting and construction of angles.

Bisecting and Construction of Angles

To bisect a given angle means to divide it into two equal parts.

**Procedure**

a. With the compass pin placed on O, the vertex of the angle, draw arcs on the arms of the angle with the same radius.

b. The point of intersection of the arcs on the arms of the angle are labelled A and B.

c. Above the angle that is to be bisected, draw two intersecting arcs using the same radius with their centres (where the compass pin is placed) being the points A and B.

d. Label the point of intersection of the arcs R.

e. Join O and R. this line OR divides angle ‘a’ into two equal halves.
Constructing an angle of 90° and 45°

a. Draw a horizontal line PQ as shown below.

b. Using a convenient radius, place the compass pin on the line at a point, say, O and describe a semicircle to intersect with the line PQ at points A and B.

c. Using any convenient radius step at A and B and draw arcs to intersect at C.

d. Join O to C to obtain angle 90°.

e. Bisect the angle 90° to obtain 45° as shown below.

![Diagram showing construction of 90° and 45° angles]

Constructing an angle of 120°, 60°, 30° and 15°

Procedure

a. Draw a straight line and mark a point O, the vertex of the angle to be constructed.

b. With the compass pin on O, draw a semicircle and mark its ends on the line as A and B. Angle AOB is a straight angle (180°) whose arms are OA and OB.

c. With the same radius, put the compass pin on any of A or B (say B) and draw an arc on the semicircle. Label this point C.

d. Draw a line through O and C. Angle BOC=60°, automatically angle AOC = 120° (60° and 120° are pair of supplementary angles and they share a common arm OC)

e. Bisect angle BOC = 30°

f. Bisect angle BOL= 15°
Activities

1. Explain how to construct a given angle
2. A student claims that an angle of $65^\circ$ can be constructed. Explain why you think the student is correct or not correct.
3. Construct the following angles: $45^\circ$, $90^\circ$.
4. Construct angles of $60^\circ$, $30^\circ$ and $15^\circ$.

Summary

In this section, we have discussed the bisection of a line and angles and construction of basic angles ($60^\circ$, $90^\circ$, $30^\circ$, $15^\circ$, $45^\circ$). I urge you to try your hands on these concepts for better understanding. Good luck in your attempt.
SECTION FOUR: CONSTRUCTION OF OTHER ANGLES

For any angle to be constructed, by the use of a pair of compasses and a ruler, a straight angle (180°) is the first to construct.

Two angles that sum up to 180° are called supplementary angles and any pair of supplementary angles can be constructed the same way. The two angles share the one common arm and their other arms are the different arms of the straight angle.

Procedure

a. Draw a straight line and mark a point O, the vertex of the angle to be constructed.
b. With the compass pin on O, draw a semicircle and mark its ends on the line as A and B. Angle BOC is a straight angle (180°) whose arms is OA.
c. With the same radius, put the compass pin on any of B or C (say C) and draw an arc on the semicircle. Label this point A.
d. Draw a line through O and A. \( \angle AOC = 60° \), automatically \( \angle BOA = 120° \) (60° and 120° are pair of supplementary angles and they share a common arm OA)

Constructing an angle of 135°

a. To obtain an angle of 135°, construct angle 90°.
b. Bisect the remaining 90° on the left to obtain 45° and add it to 90° to obtain the 135 as shown below.
Constructing an angle of 75°

a. To construct an angle of 75° at point O, bisect the angle between the 60° and 90° lines (60° +15°) =75°.

b. Trisect a straight angle (180°) into three (60°s).

c. Bisect the middle 60° into two (30 °s). i.e. ∠POD = ∠DOQ = 30°.

d. Note that this same bisector also bisects the straight angle into (90°s). ∠AOC = ∠BOD

e. Bisect any of the two (30°s) into two (15°s). ∠AOC = ∠TOQ = 15°

f. ∠BOT = 75°. i.e ∠AOD + ∠DOT
Constructing an angle of 150°

a. Construct angle 120°

b. Bisect the remaining angle on the left of the 120° (i.e 60°) to obtain angle 30° to be added to the 120° to obtain 150° as shown below.(refer to construction of angle 30°)

Activities

1. Explain how to construct a given angle

2. A student claims that an angle of 65° can be constructed. Explain why you think the student is correct or not correct.

3. How will you construct angles of 75°, 105°, 120° and 135°?
Summary

In this section you have learnt how to construct other angles such as 75°, 105°, 135°, 150° etc. We went through the various procedures of constructing these angles. I am sure you will practice and master the construction of these and other angles. Well done!
SECTION FIVE: TEACHING CONSTRUCTION OF TRIANGLES

I am pleased to welcome you to section 5 of Unit 2. How have you been doing? You are doing well in your study and activities? In this section, we are going to examine the construction of triangles. I am sure this concept is not new to you. You must have come across it many times in your academic pursuit.

**Construction of Triangles**

The construction of a triangle assumes any of the three forms below:

a. The lengths of all the three sides are known
b. Two angles and the length of the side are known
c. The lengths of two sides and an angle are known

The construction of a triangle with the length of all the three sides (given)

**Worked Example**

Construct \( \triangle ABC \) given that

\[ |AB| = 8 \text{cm}, \quad |BC| = 10 \text{cm} \text{ and } |AC| = 11 \text{cm}. \]

**Procedure**

a. Construct line AB of length 8cm
b. With centre A and radius 12cm, draw an arc above the line AB.
c. With centre B and radius 10cm, draw an arc to intersect the first one at C.
d. Join A to C and B to C
e. \( \triangle ABC \) is the triangle required
Note: You can choose to start with any of the three given sides.

Construction of a triangle with the length of two sides and an interior angle given

Worked Example

Construct $\triangle PQR$ with $\angle RQP = 60^\circ$, $|PQ| = 10cm$ and $|PR| = 12cm$

Procedure

a. Construct a line PQ of length 10cm.
b. Construct an angle of $60^\circ$ at P
c. With the aid of your pair of compasses, measure 12cm and make an arc on the line that makes an angle of $60^\circ$ with PQ.
d. Join Q to R
e. PQR is the required triangle
Construction of a triangle with two interior angles and the length of a side given

Example

Construct $\triangle ABC$ with $\angle CAB = 75^\circ$, $|AB| = 12\text{cm}$ and angle $ABC = 45^\circ$.

Procedure

a. Construct a line $AB$ of length $12\text{cm}$
b. Construct angle $75^\circ$ at A
c. Construct an angle of $45^\circ$ at B
d. Extend the lines that make angles $75^\circ$ at A and $45^\circ$ at B to meet at C.
e. $ABC$ is the required triangle.
Activities

1. Construct an equilateral triangle with sides 7.2cm.

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2. Explain how to construct a triangle with the length of two sides and an interior angle given.

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3. Construct a triangle with two interior angles and the length of a side given.

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Summary

Congratulations for working through this section on the construction of triangle. We have also explain how to construct a triangle with the length of two sides and an interior angle given. Finally we had a discussion on constructing a triangle with two interior angles and the length of a side given. It is my hope that all these discussions have gone well with you. Well done!
Hello, my dear students. I welcome you to the last section of Unit 2. Our focus of the unit has been on teaching geometric construction. I am sure you have seen some appreciable level of improvement in your skills and understanding of geometric construction. Congratulations! In this sections, we will concentrate on construction of quadrilaterals. Am sure you know what quadrilaterals are? What about examples of quadrilaterals? Well done.

**Construction of Quadrilaterals**

*Let us look at how to construct some quadrilaterals*

**Construction of square**

**Worked Example**

**Construct a square ABCD of side 10cm.**

![Square Diagram]

**Procedure**

a. Construct a line segment AB of length 10cm.
b. Construct perpendiculars at A and B on the line AB

c. With centre A and a radius of 10cm, draw an arc to cut the perpendicular on A at the point D.

d. With centre B and exactly the same radius, draw an arc to cut the perpendicular on B at the point C.

e. Join C to D

f. A square has the lengths of each of the 4 sides equal.

NB: If there is no error, then the two diagonals must be equal in length.

**Construction of a Rectangle.**

**Worked Example**

Construct a triangle ABCD with $|AB| = 12\text{cm}$, and $|BC| = 10\text{cm}$

![Diagram of a rectangle with sides labeled]({})

**Procedure**

a. Construct a line segment AB of length 12cm.

b. Construct perpendicular at A and B

c. With centre A and a radius of 10cm, draw an arc to cut the perpendicular on A at point D.
d. With centre B and exactly the same radius, draw an arc to cut the perpendicular on B at point C.
e. Join C to D
f. ABCD is the required rectangle

NB: the pair of the opposite sides have the same lengths.

Construction of a parallelogram

Worked Example

Construct a parallelogram ABCD with $|AB| = 12\text{cm}$, and $|BC| = 10\text{cm}$ and angle $ABC=105^\circ$, $BAD=75^\circ$.

Procedure

a. Construct a line segment AB of length 12cm.
b. Construct angle $ABC = 105^\circ$
c. With centre B and radius 10cm, draw an arc to cut the line forming the angle $75^\circ$ at D.
d. Join C to D

Loci
A locus is the set of all possible positions of a point, which varies according to certain conditions. The plural of locus is loci. All the points on these positions have a common property/properties.

**Locus of points Equidistant from a fixed point**

This locus is a circle whose centre is a fixed point with a given radius (the equal distance).

**Worked Example**

A point moves in a plane so that its distance from a fixed point is 3cm. Construct this locus.

**Solution**

a. Open your compasses to a radius of 3cm
b. Place it at a point (mark the point as O) and construct a circle with the point O as the centre.
Locus of a points equidistant from two fixed points.

This locus is the perpendicular bisector of the line joining two given points. All the points on this line have common property. Let \( P \) be any variable point on this line (locus). If the two points are \( Q \) and \( R \) then \( |PQ| = |PR| \).

**Worked Example 1**

\[
\begin{align*}
|P_1Q| &= |P_1R| \\
|P_2Q| &= |P_2R| \\
|P_3Q| &= |P_3R|
\end{align*}
\]

**Worked Example 2**

A point moves in a plane such that it is always equidistant from two given points \( A \) and \( B \), which are a fixed distance of 8cm apart. Describe briefly and construct its locus.

Solution

The locus of the point will be the perpendicular bisector of the line joining \( A \) and \( B \).
Locus of points equidistant from two intersecting straight lines

The locus is also the angle bisector of the angle between the two intersecting straight lines.

Let OT and OQ be two lines intersecting at O. the locus is the bisector of angle TOQ. Every point on this bisector has a common property. If P is a variable point on the bisector, then the length of the perpendicular from P to OT, is the same as the length of the perpendicular from P to OQ.

Worked Example 3

If point P, moves in a plane such that its distance from two given intersecting straight lines AB and AC is equidistant, what is its locus?

Solution

The locus of P is the bisector of the angle between AB and AC.
Locus of points equidistant from two parallel lines.

This locus is the third parallel line which is midway the two given parallel lines.

If $AB$ is parallel to $CD$, then the third line indicating the locus is parallel to both and is half way the perpendicular distance between them.

**Worked Example**

A point moves in a plane such that its distance from two given parallel lines is constant. Given that the distance between the parallel lines is 12cm, describe and construct the locus of the point.

**Solution**

Mark two points on one of the two parallel lines as $A$ and $B$.

a. From points $A$ and $B$, draw perpendicular lines to meet the other parallel line at $C$ and $D$ such that $|AC| = |BD| = 12$cm.
b. Adjust your compass such that the radius is 6cm (half of 12cm).
c. With the compass pin at A, draw an arc on AC and name it P_1
d. Repeat the process with compass pin at B and mark on BD a point P_2 6cm from B.
e. Join P_1 and P_2. This is the required line (locus) which is equidistant from AB and CD.

![Diagram](image)

Activity 1

a. Using a pair of compasses only,
i. construct a parallelogram ABCD with │AB│=8cm, angle BAC=45˚ and angle ABC=60˚.
ii. locate a point P inside triangle ABC such that │PA│=│PB│ and │PC│=4cm

b. Measure │PD│

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Activity 2

Using a pair of compasses only, construct:

1. Explain quadrilaterals and give at least three examples.

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2. a quadrilateral ABCD such that $|AB|=8\text{cm}$, $|BC|=|AD|=6.5\text{cm}$, $ABC=60^\circ$ and angle $BAD=75^\circ$. Construct the locus $l_1$ of points equidistant from A and B. Construct the locus $l_2$ of points equidistant from B and C. Locate a point P where P is the point of intersection of $l_1$ and $l_2$.

3. a parallelogram with sides 6cm and 9cm, the angles between these sides being $60^\circ$. Measure the diagonals.

4. a rectangle measuring 7.4cm by 10.3cm and measure the length of the diagonal.

Summary

I hope you have by this time realized that to construct a shape or a diagram means to draw it accurately. The most accurate way to construct a diagram is to use compasses and a ruler only. Note that it is not advisable to draw lines which are longer than the required segments, using a ruler or a pair of compasses to mark off the required points. Read more about construction of quadrilaterals so that you can improve upon your construction skills. Congratulations!
UNIT 3
TRIGONOMETRY, VECTORS AND BEARINGS

Introduction

Dear student, you are welcome to the third unit of this course. The first three sections of the unit will introduce you to basic trigonometry. It covers concepts such as right-angle triangles, Pythagorean triples and basic trigonometric ratios (sine, cosine and tangent). You are encouraged to use your prior learning of Pythagoras rule, measure of angles in various quadrants of a circle and right-angle triangles to help you achieve the section objectives. We know after completing the entire unit, you will come to love these concepts and accept that indeed they are relevant concepts with real-life applications.

Unit Content

In this unit, we will explore the following concepts in six sections;

Section 1: Angles
Section 2: Right-angled Triangles
Section 3: Trigonometry Ratios and their Application to real life situations
Section 4: Algebra of Vectors and Vector Notations
Section 5: Vector Operations
Section 6: Magnitude and Direction of Vectors

SECTION 1: ANGLES

Dear student, consider a situation where we have to measure the height of a skyscraper without necessarily climbing it or to measure the height of an electric pole in front of a lecture block. What about measuring the height of the faculty block? Imagine a boy wants to swim across a fast-flowing river. If he swims directly across, he will be swept downstream away by the high current. In your view, where should he aim at so that he can safely swim to the opposite ends of the river? The solution in all these scenarios involve the application of the concept of trigonometry. These therefore makes the learning of trigonometry imperative. This section focuses on trigonometric angles as they are foundational to the whole concept of trigonometry.
Section objectives

By the end of this section, you will be able to;

- Explain angles
- Identify various kinds of angles
- Explain angular measures in degrees and radians and their relationships
- Use the three basic trigonometric ratios to determine angular values in degrees in the quadrants of a circle

We will take a brief insight into where the concepts of trigonometry came from, how they were discovered, and how their uses in the past relate to how they are currently used and taught. This will provide you with the extra understanding you need to put these concepts to use, be it as measuring devices, or as functions. With this better insight, one would be able to see the value in studying trigonometry as a component of mathematics, instead of a detached unit from the subject.

The term “trigonometry,” although not of native Greek origin, it derives its name from the Greek words trigonon, meaning “triangle,” and metria, meaning “measurement.” It means “measurement of a triangle”. As the name implies, trigonometry ultimately developed from the study of right triangles by applying the relationships between the measures of its sides and angles to the study of similar triangles. Trigonometry is concerned with the relationship between the size of the angles and the length of sides of a plane figure such as triangles. Therefore, we can conclude that trigonometry is that branch of mathematics concerned with the measurement of sides and angle of a plane triangle and the investigations of the various relations which exist among them and their applications. Hipparchus of Nicaea is the known figure in trigonometry. He is regarded as the father of trigonometry though other scholars made contributions in the field.

ANGLES

An angle is simply the degree of turn from a fixed point. It is also defined as the measure of the rotation of a line from one position to another about a fixed point on it.

An angle is also formed by rotating a ray around its endpoints. The two rays are known as the initial side and the terminal side. These two rays form the sides of the angle. The endpoint then becomes the vertex of the angle.
In figure 3.1, the first position OX is called initial line (position) and second position OP is called terminal line or generating line(position) of XOP. If the terminal side resolves in anticlockwise direction the angle described is positive as shown. If terminal side resolves in clockwise direction, the angle described is negative.

**Angle Rotations**

**Positive Angles**

An angle, becomes a positive angle if it is generated by a counter clockwise rotation about the origin.

**Negative Angles**

An angle is a negative angle if it is generated by a clockwise rotation about its origin.

**Standard angle**

An angle is described as being in standard position when the starting side (initial side) coincides with the positive x-axis with its vertex at the origin.

**Types of angles**

Angles can take one of the following forms;

(a) An angle is **acute** if it is between $0^\circ$ and $90^\circ$. Thus, $0^\circ<\theta<90^\circ$

(b) An angle is a **right angle** if it equals $90^\circ$. Thus, $0^\circ=\theta=90^\circ$

(c) An angle is **obtuse** if it is between $90^\circ$ and $180^\circ$. Thus, $90^\circ<\theta<180^\circ$

(d) An angle is a **straight angle** if it equals $180^\circ$. Thus, $\theta=180^\circ$
e) An angle is a **reflex angle** if it is between $180^\circ$ and $360^\circ$. Thus, $180^\circ < \theta < 360^\circ$

Two positive angles can also sum up to form a right angle, thus $90^\circ$. Such angles are said to be **complementary** angles while two positive angles that sum up to form a straight line, thus $180^\circ$ are said to be **supplementary** angles.

**Degree Measurement of Angles**

Just as we measure time in seconds, minutes and hours, angles are measured in degrees. This unit of measurement was developed by the Babylonians about 4000 years ago. The measurement of angles in degrees is guided by these principles;

- 360 degrees shows a complete rotation about a fixed point
- 90 degrees shows one fourth of a complete rotation
- 180 degrees shows one half of a complete rotation

$360^\circ$ degrees is a complete revolution. 1/60 degrees is a minute and 1/60 minute is a second.
Example: $45^\circ 7' 12''$ is read as forty-five degrees, 7 minutes and 12 seconds.

Which when interpreted is $45^\circ + 7/60^\circ + 12/3600^\circ$

**Radian Measurement of Angles**

Angles can also be measured in radians. A radian is an angle formed by an arc which has the same length as the two radii that subtend it. In other words, the central angle that is subtended by an arc with same length as the radii is called a *radian*. The symbol for radian is rad. A radian can therefore be considered as the measure of the central angle whose length is the same as the radii of the circle. If so, the number of radians in a complete revolution is $2\pi r/r$. This is equal to $2\pi$ rad. When this is equated to $360^\circ$, it gives the value of radians in degrees.

$2\pi \text{ rad} = 360^\circ$. Therefore, a radian has an angle equal to $180/\pi$. A radian is approximately $57^\circ$.

In general, if the length of arc, $s$ units and the radius is $r$ units, then

\[
\theta = \frac{s}{r}
\]

That is the size of the angle ($\theta$) is given by the ratio of the arc length to the length of the radius.

For example:

If $s = 3 \text{ cm}$ and $r = 2 \text{ cm}$, then

\[
\theta = \frac{s}{r} = \frac{3 \text{ cm}}{2 \text{ cm}} = 1.5 \text{ radians}
\]

Source: www.Learnalberta.ca

In trigonometry, angles are measured in an anticlockwise direction from the positive x-axis from $0^\circ$ to $360^\circ$ using trigonometric ratios. What are these trigonometric ratios? The trigonometric ratios are of three basic types. These are;

i. Sine (Sin)

ii. Cosine (Cos)

iii. Tangent (Tan)

These ratios may be positive or negative depending on the quadrant in which it lies. The sign of the ratio in a particular quadrant can be remembered from the diagram below;
From the diagram,

In the 1\textsuperscript{st} quadrant (0\textdegree-90\textdegree) all the ratios are positive

In the 2\textsuperscript{nd} quadrant, (90\textdegree-180\textdegree) only sine is positive

In the 3\textsuperscript{rd} quadrant (180\textdegree-270\textdegree) only tan is positive

In the 4\textsuperscript{th} quadrant (270\textdegree-360\textdegree) only cos is positive

\textbf{Quadrants:}

Two mutually perpendicular straight lines divide the plane into four equal parts, each part is called quadrant. Thus, there is the 1\textsuperscript{st}, 2\textsuperscript{nd}, 3\textsuperscript{rd} and 4\textsuperscript{th} quadrants respectively.

In first quadrant, the angle varies from 0\textdegree to 90\textdegree in anti-clockwise direction and from –270\textdegree to –360\textdegree in clockwise direction.

In second quadrant, the angle varies from 90\textdegree to 180\textdegree in anti-clockwise direction and –180\textdegree to –270\textdegree in clockwise direction.

In third quadrant, the angle varies from 180\textdegree to 270\textdegree in anticlockwise direction and from –90\textdegree to –180\textdegree in clockwise direction.

In fourth quadrant the angle Fig.3.4 vary from 270\textdegree to 360\textdegree in anticlockwise direction and from 0\textdegree to –90\textdegree in clockwise direction.
Activity 1

1. Determine the type of angle with the following measurement;
   a) 56°  b) 107°  c) 180°  d) 278°  e) 349°  f) 360°

2. Find the measure of each angle as represented by the letters below;

   \[a) \ \angle m = 280°\]
   \[b) \ \angle 3x°\]
   \[c) \ \angle 3x° = 20°\]

3. Using your answers in (2), determine the type of angle for each letter.

Suggested answers

1. a) Acute  b) obtuse  c) straight line  d) reflex  e) reflex  f) circle

2. a) M=75  b) a=10  c) x=40

3. m= acute angle  b) acute angle  c) acute angle

Summary

In a nut shell, we have learnt that;

- The term “trigonometry” comes from the Greek word *trigonon*, meaning “triangle,” and the Greek word -*metria*, meaning “measurement.”
- Trigonometry therefore means the “measurement of a triangle”
- Hipparchus of Nicaea is the father of trigonometry
• An angle is formed by rotating a ray about a fixed point.
• Two positive angles can also sum up to form a right angle, thus 90°. Such angles are said to be complementary angles while two positive angles that sum up to form a straight line, thus 180° are said to be supplementary angles.
• An angle, becomes a positive angle if it is generated by a counter clockwise rotation about the origin.
• Also, an angle is a negative angle if it is generated by a clockwise rotation about its origin.
• An angle is described as being in standard position when the starting side (initial side) coincides with the positive x-axis with its vertex at the origin.
• Angles can be either acute, obtuse, straight, right or reflex depending on its measure in degrees.
• Angles are usually measured in degrees and radians.
• The central angle that is subtended by an arc with same length as the radii is called a radian.
• The symbol for radian is rad.
• The use of degree as a unit of measurement of angles was developed by the Babylonians about 4000 years ago.
• Angles may form within four main quadrants, 1\textsuperscript{st}, 2\textsuperscript{nd}, 3\textsuperscript{rd} or 4\textsuperscript{th} and that determines whether it will be negative or positive.
Chapter 1 SECTION 2: RIGHT ANGLE TRIANGLES

Introduction

Dear students, we want to continue our study of trigonometry in this section by looking at triangles, and for a while we will consider only right triangles. Once we have understood right triangles, we will know a lot about other triangles as well. It is worth knowing that there are some triangles that have no right angles. Such triangles are known as the oblique triangles. They usually contain either three acute angles or two acute and one obtuse angle. Take note of these triangles to help you differentiate them from what our discussion today will be about.

Section objectives

By the end of this section, you should be able to;

- Describe a right-angle triangle and its sides
- Derive the Pythagoras theorem from right-angle triangles
- Determine the unknown sides of a right-angle triangle using the Pythagoras theorem
- Apply Pythagoras theorem to solve right-angled related problems
- Identify sets of numbers that form the Pythagoras triple
- Use the multipurpose multiplication chart to explore the relationship among the Pythagorean triples
- Prove whether a set of numbers form the Pythagoras triple or not

RIGHT ANGLE TRIANGLE

I believe you have come across the word, ‘right angle’ in our previous section. We learnt that such angles are angles that measure up to 90°. A right-angle triangle is therefore any triangle which has one of its angles measuring 90° (right-angle). In any right-angled triangle, it is possible to determine the sides of any of the acute angles once the other is known. The side of a right-angled triangle are given special names that makes it easier to find the trigonometry ratio. These names are;

i. Hypotenuse
ii. Adjacent
iii. Opposite
A fundamental property of right-angled triangles is that Pythagoras’ Theorem holds. This triangle has three sides; a longest side which is always called the hypotenuse, a side of length $x$ and one of length $y$. You also have three angles; the right-angle itself (depicted by the small square) and two others given by $\theta$ (theta) and $\phi$ (phi). If you look carefully you can see that the hypotenuse is the side that is opposite the right angle – in a way the hypotenuse plays no part in the right-angle which is defined by the join between sides $x$ and $y$.

The hypotenuse is always opposite the right angle in a right-angled triangle. Definitions of adjacent and opposite sides are different for the angle $\theta$. Can you see that the side $x$ is opposite the angle $\theta$ and that side $y$ is adjacent to $\theta$? This is the crucial point here, when you are defining the opposite and adjacent sides of an angle, the sides you choose depend on the angle you are interested in. Moreover, the hypotenuse always remains the longest side, the one opposite the right-angle. This idea of sides which are opposite and adjacent with respect to certain angles is essential in defining the trigonometric ratios. The longer of the sides ($x$ or $y$) is always opposite the larger of the angles ($\theta$).
If we know the length of two sides of a 90° triangle, we can determine the angular measure of the reference angle $\theta$. The reference angle determines which side we refer to as opposite and which is referred to as adjacent. Wherever angle $\theta$ is, this is the reference point. The side opposite of $\Theta$ is simply referred to as the “side opposite” and the side connected to $\theta$ is called “side adjacent”

**Consider the diagrams below;**

![Diagram](image)

a) the side PQ, opposite the right-angle R, is the hypotenuse.

PR is the side adjacent to angle P

RQ is the side opposite to angle P

b) The side PQ is still the hypotenuse

RP is now the side opposite to angle Q

RQ is now the side adjacent to angle Q

The idea of right-angled triangles brought about the Pythagoras’ theorem. The Pythagorean theorem is named after the Greek philosopher Pythagoras, though it was known well before his time in different parts of the world such as the Middle East and China. The Pythagorean theorem is correctly stated in the following way. Pythagoras’ theorem states that for any right-angle triangle:

**Square of the hypotenuse = sum of the squares of the other two sides.**

From figure 1 below

$$|BC|^2 = |AB|^2 + |AC|^2$$

$$c^2 = a^2 + b^2$$

From $c^2 = a^2 + b^2$ we can also write the following;
$$a^2 = c^2 - b^2 \quad \text{OR} \quad a = \sqrt{c^2 - b^2}$$

$$b^2 = c^2 - a^2 \quad \text{OR} \quad b = \sqrt{c^2 - a^2}$$

NB: the longest side or the side opposite to the right angle is always the hypotenuse

Example 1

Find $|PQ|$ in the figure below;

Solution

By Pythagoras’ theorem;

$|PQ|^2 = |RP|^2 + |QR|^2$

$= 12^2 + 5^2$

$= 144 + 25$

$= 169$

$|PQ|^2 = 169$

$|PQ| = \sqrt{169}$

$= 13 \text{ cm}$

Example 2

Find the length of AC in the figure below;
Solution

By Pythagoras’ theorem;

\[ |BC|^2 = |AB|^2 + |AC|^2 \]

\[ 5^2 = 3^2 + |AC|^2 \]

\[ 25 = 9 + |AC|^2 \]

\[ |AC|^2 = 25 - 9 \]

\[ |AC|^2 = 16 \]

\[ |AC| = \sqrt{16} \]

\[ |AC| = 4 \text{ cm} \]

Example 3

Find \(|PT|\) in the figure below

Solution

\[ |QT|^2 = |PT|^2 + |QP|^2 \]

\[ 17^2 = |PT|^2 + 15^2 \]
\[ |PT|^2 = 289 - 225 = 64 \]
\[ |PT| = \sqrt{64} = 8 \text{ cm} \]

**Example 4**

In the triangle ABC below, \(|AB| = |BC| = 5\text{ cm} \) and \(|AC| = 8\text{ cm} \). find \(|BD|\)

![Diagram](image)

**Solution**

From the right-angle triangle ABD

\[ |AB|^2 = |AD|^2 + |BD|^2 \]
\[ |5|^2 = |4|^2 + |BD|^2 \]
Since \(|AD| = \frac{1}{2} \text{ of } |AC| , |AD| = \frac{1}{2} \text{ of } 8 = 4 \]
\[ |5|^2 = |4|^2 + |BD|^2 \]
\[ |BD|^2 = 5^2 - 4^2 = 25 - 16 = 9 \]
\[ |BD|^2 = 9 \]
\[ |BD| = \sqrt{9} = 3 \text{ cm} \]

**Example 5**

In the diagram, \( \angle PQR = \angle PRS = 90^\circ , |PS| = 13\text{ cm} , |PQ| = 3\text{ cm} \) and \(|QR| = 4\text{ cm} \). find \(|RS|\)
Solution

By Pythagoras’ theorem;

\[ |PR|^2 = |PQ|^2 + |QR|^2 \]
\[ |PR|^2 = 3^2 + 4^2 = 9 + 16 = 25 \]
\[ |PR| = \sqrt{25} = 5 \text{ cm} \]

Also, from the right-angled triangle PRS

\[ |PS|^2 = |PR|^2 + |RS|^2 \]
\[ 13^2 = 5^2 + |RS|^2 \]
\[ 169 - 25 = |RS|^2 \]
\[ |RS|^2 = 144 \]
\[ |RS| = \sqrt{144} = 12 \text{ cm} \]

EXAMPLE 6

Show that a triangle with sides 3, 4, and 5 is a Right angled-triangle

Solution

We can apply statement to see if it is a right triangle. In fact, \( 5^2 = 25 = 3^2 + 4^2 \),

so, the angle opposite the side of length 5 is a right angle.

NB: that we can use statement of the Pythagorean theorem to solve this problem.
APPLICATION OF PYTHAGORAS THEOREM

Problems

1. A ladder leans against a vertical wall of height 12m. If the foot of the ladder is 5m away from the wall, calculate the length of the ladder.
2. A ladder of 4.5 m long. The foot of the ladder is 2m away from the base of the wall. How far up the wall is the top of the ladder?

Suggested Answers

1. 13m long
2. 4.03 (2 d.p) up the wall

Check Your Progress

1. A) In the quadrilateral ABCD below, |AB| = 3cm, |BC|=4cm, |CD| = 12cm and angle ABC = 90° and ACD=90°.

   ![Diagram]

   Calculate
   i. The perimeter of ABCD
   ii. The area of ABCD

2. Find the value of the letters in the figure below;
   i.
3. An isosceles triangle has equal sides 6cm long and a base 4cm long. Find the altitude of the triangle, leaving your answer in surd form.

4. The sides of a rectangular floor are x m and (x+7) m. the diagonal is (x+8) m. calculate
   i. The value of x
   ii. The area of the floor

5. A rectangular box has length 10 cm, breadth 9cm and 6cm high. What is the longest length of the longest stick that will fit into the box?

6. Calculate $|PR|$ from the diagram below

7. Find $|PR|$ and $|SR|$ correct to a decimal place.

Suggested answers
1. i. 32cm
   ii. 13cm
2. i. 15.68cm
3. \(4\sqrt{2}\)

4. i. 5m   ii. 60m\(^2\)

5. 14.73m

6. 16cm

7. \(|PR| = 12.65\text{ cm}\) and \(|SR| = 4\text{ cm}\)

**Pythagorean Triple**

Sometimes, the three lengths of a right-angled triangle are all whole numbers. When this occurs, they are called a Pythagorean Triple or Triad. The most commonly known of these is (3,4,5) representing the triangle.

Pythagorean triple is a set of three integers which obey the Pythagoras theorem. Thus, the sum of the squares of two of the sides of a right-angled triangle equals the square of the third side.

For any two positive integers \(a\) and \(b\), where \(a > b\), the three sides of a right-angle triangle can be expressed in terms of \(a\) and \(b\) to generate the Pythagorean triplets as shown below:

\[
\left\{ \frac{1}{2} \left( a^2 - 1 \right), \frac{1}{2} \left( a^2 + 1 \right) \right\}
\]

However, if only one side of the right-angled triangle is known, then the Pythagorean triple can be generated using the sides.

\[
\left\{ \frac{1}{2} \left( a^2 + 1 \right) \right\}^2 \equiv \left\{ \frac{1}{2} \left( a^2 - 1 \right) \right\}^2 + a^2 \text{ where } a \text{ is an odd integer greater than 1.}
\]

**Example of numbers that form the Pythagorean Triplets are**
Other Triads can be based on multiples of the base Triads; the Triads (6,8,10), (9,12,15), (12,16,20) ... are based on the base Triad (3,4,5).

This means that to get other triplets, multiply each of (3,4,5) by a whole number.

For instance, (3×2=6), (4×2=8) and (5×2=10) hence the next triplets are (6, 8 and 10)

Using the multi-purpose multiplication chart

In this chart, the top row numbers are multiplied by the left column numbers to generate each of the product (numbers) in the cell as shown below;

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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<td>25</td>
<td>30</td>
</tr>
<tr>
<td>6</td>
<td>6</td>
<td>12</td>
<td>18</td>
<td>24</td>
<td>30</td>
<td>36</td>
</tr>
</tbody>
</table>

Keys

Each of the colored columns of the chart represent a set of Pythagorean triples.

For instance;

The set for Yellow {3,4 and 5} = 3² + 4² = 5²
The set for Red \{6,8 and 10\} \quad =6^2 \quad + \quad 8^2 \quad = \quad 10^2

The set for Green \{9,12 and 15\} \quad =9^2 \quad + \quad 12^2 \quad = \quad 15^2 \quad , \quad \text{etc.}

**Proof**

For these numbers to be Pythagorean triplets, then the sum of the squares of two of the sides must equal the square of the third side;

Thus

\[ c^2 = a^2 + b^2 \]

Or

\[ a^2 + b^2 = c^2 \]

For \[3, 4 \text{ and } 5\]

\[ a^2 + b^2 = c^2 \]

Where \(a=3, b=4\) and \(c=5\)

\[ 3^2 + 4^2 = 5^2 \]

\[ 9 \quad + \quad 16 \quad = \quad 25 \]

\[ 25 \quad = \quad 25 \quad \text{VALID} \]

Hence, the numbers \(3, 4 \text{ and } 5\) are Pythagorean triplets.

Example 1

Find the Pythagorean triplets if \(x=3\) and \(y=2\)

**Solution**

\[ x^2 = 3^2 = 9 \]

\[ y^2 = 2^2 = 4 \]

\[ x^2 - y^2 = 9 - 4 = 5 \]

\[ 2xy = 2 (3\times2) = 12 \]
\[ x^2 + y^2 = 9 + 4 = 13 \]

Therefore, the Pythagorean triples are \{5, 12 and 13\}

Example 2

The sides of a right-angled triangle are \((x-1)\) cm, \((x+3)\) cm and \((x+7)\) cm. Find the value of \(x\)

**Solution**

\[(x + 7)^2 = (x + 3)^2 + (x - 1)^2\]

\[x^2 + 14x + 49 = x^2 + 6x + 9 + x^2 - 2x + 1\]

\[x^2 - 10x - 39 = 0\]

\[x^2 - 13x + 3x - 39 = 0\]

\[x(x-13) + 3(x-13) = 0\]

\[(x+3)(x-13) = 0\]

\[(x+3) = 0\]

\[X = -3\]

\[(x-13) = 0\]

\[X = 13\]

Hence \(x = 13\)

Example 3

The diagonals of a rhombus are 8cm and 10cm long. Calculate correct to one decimal place the length of side of the rhombus.

**Solution**
Let the vertices of the rhombus be ABCD

By Pythagoras’ theorem;

\[ |DC|^2 = |OC|^2 + |OD|^2 \]
\[ |DC|^2 = 5^2 + 4^2 \]
\[ |PR|^2 = 25 + 16 = 41 \]
\[ |PR| = \sqrt{41} = 6.4 \]

Check Your Progress

1. Prove to show whether the following set of numbers are Pythagorean triplets or Not;
   i. [9, 40 and 41]
   ii. [8, 15 and 17]
   iii. [7, 24 and 25]
   iv. [6, 8 and 11]
   v. [5, 12 and 13]
   vi. [3, 4 and 5]
   vii. [15, 36 and 40]
   viii. [4, 5 and 10]
2. Generate Pythagorean triplets using the following pairs of positive integers.
   a) 3 and 1
   b) 4 and 1
   c) 4 and 2
   d) 5 and 2

3. If three sides of a triangle are given as \( x^2 - y^2 \), \( 2xy \) and \( x^2 + y^2 \). Show that the triangle is a right angles triangle.

Suggested answers

1. i. Yes  ii. Yes  iii. Yes  iv. No
   v. Yes  vi. Yes  vii. No  viii. No

2. a) \{6, 8 and 10\}
   b) \{15, 8 and 17\}
   c) \{12, 16 and 20\}
   d) \{20, 21 and 29\}

3. Clue: \( x^2 + y^2 = x^2 - y^2 + 2xy \)

Summary

As a recap, we learnt in this section that;

- A right-angled triangle is therefore any triangle which has one of its angles measuring 90° (right-angle)
- This triangle has three sides; a longest side which is always called the hypotenuse, the adjacent and the opposite.
- The hypotenuse is always opposite the right angle in a right-angled triangle.
- There are some other triangles that may not have right angles. Such triangles are known as the oblique triangles.
- Oblique triangles usually contain either three acute angles or two acute and one obtuse angle.
- A fundamental property of right-angled triangles is that Pythagoras’ Theorem holds
- The Pythagorean theorem is named after the Greek philosopher Pythagoras
- Pythagoras’ theorem states that for any right-angled triangle: Square of the hypotenuse is equal to the sum of the squares of the other two sides.
- We can apply the theorem of Pythagoras to determine the unknown sides of a right-angle triangle.
- Pythagorean triple is a set of three integers which obey the Pythagoras theorem.
• For any three sides of a right triangle to be a Pythagorean triple, the sum of the squares of two of the sides of a right-angled triangle must be equal to the square of the third side.
• The most commonly known of the Pythagorean triple is (3,4,5) representing a triangle.
• Other Triads can be based on multiples of the base Triads; the Triads (6,8,10), (9,12,15), (12,16,20) ... are based on the Triad (3,4,5).
• This means that to get other triplets, multiply each of (3, 4, 5) by a whole number. For instance, (3×2=6), (4×2=8) and (5×2=10) hence the next triplets are (6, 8 and 10)
• For any three numbers to be Pythagorean triplets, then the sum of the squares of two of the sides must equal the square of the third side; Thus, \( c^2 = a^2 + b^2 \)
Or \( a^2 + b^2 = c^2 \)
Chapter 2 SECTION 3: TRIGONOMETRIC RATIO AND ITS APPLICATION

Dear students, you are warmly welcome to section 3 of unit 3 of the course. In this section, we will take a closer look at the trigonometric ratios that was mentioned in the beginning sections of this unit. We will also establish the relationship among the three basic trigonometric ratios and how they are applied to real life situations such as finding the height of a vertical object without measuring, through investigations.

Section objectives

By the end of this section, you will be able to;

- Explore derivative algorithm for sine, cosine and tangent
- Use simple mnemonics to learn the rules of the basic trigonometric ratios
- Find the values of trigonometric ratios (sine, cosine and tangent)
- Find size of angles using trigonometric ratios.
- Solve right angled triangle problems using trigonometry and its related concepts
- Find the relationship among the three trigonometric ratios
- Pose and solve word problems based on application of trigonometry ratios.

TRIGONOMETRIC RATIO

The trigonometric ratios are of three basic types.

These are;

i. Sine (Sin)
ii. Cosine (Cos)
iii. Tangent (Tan)

These ratios may be positive or negative depending on the quadrant in which it lies.

The Sine of an Angle

In a right-angled triangle the sine of an angle is defined as: length of side opposite to the angle/length of hypotenuse. As the hypotenuse in a triangle is always the longest side, the value for the sine of an angle is never bigger than 1. In mathematical writing, the word “sine” is shortened to “sin” and the angle of interest is written after. For example, sinθ. There is no multiplication here and the text “sinθ” implies the sine of the angle θ.

The abbreviation sin is usually used for Sine.

\[
\text{Sine of an angle} = \frac{\text{side opposite to the angle}}{\text{hypotenuse}}
\]
The simplest way to remember this is

Sine of an angle = SOH

Where

S = Sine of an angle
O = Opposite
H = Hypotenuse

$$\text{Sine of an angle} = \text{SOH} = \frac{\text{side opposite to the angle}}{\text{hypotenuse}}$$

From the diagram above

Sin P = \frac{|RQ|}{|PQ|}

Sin Q = \frac{|RP|}{|PQ|}

**Definition:** For any acute angle $\alpha$, we draw a right triangle that includes $\alpha$. The sine of $\alpha$, abbreviated sin $\alpha$, is the ratio of the length of the leg opposite this angle to the length of the hypotenuse of the triangle.

For Example, in the right triangle $ABC$ (diagram above), $\sin \alpha = \frac{a}{c}$.

We can see immediately that this definition has a weak point: it does not tell us exactly which right triangle to draw. There are many right triangles, large ones and small ones that include a
given angle \(a\).
Let us try to answer the following questions.

Example 1: Find \(\sin 30^\circ\).

**feedback**

Formally, we are not obliged to solve the problem, since we are given only the measure of the angle, without a right triangle that includes it.

**Solution:** Draw some right triangle with a \(30^\circ\) angle:

\[
\begin{align*}
C &= 20 \\
A &= 10 \\
\end{align*}
\]

For Example, we might let the length of the hypotenuse be 20. Then the length of the side opposite the \(30^\circ\) angle measures 10 units. So

\[
\sin 30^\circ = \frac{10}{20} = \frac{1}{2} = 0.5
\]

We know, from geometry, that whatever the value of the hypotenuse, the side opposite the \(30^\circ\) angle will be half this value. So, \(\sin 30^\circ\) will always be 1/2. This value depends only on the measure of the angle, and not on the lengths of the sides of the particular triangle we used.

**The Cosine of an Angle**

The abbreviation \(\text{Cos}\) is usually used to represent cosine.

Cosine of an angle = \(\frac{\text{side adjacent to the angle}}{\text{hypotenuse}}\)

The simplest way to remember this is

Cosine of an angle = CAH
Where $C = \text{Cosine of an angle}$

$A = \text{Adjacent}$

$H = \text{Hypotenuse}$

Cosine of an angle $= \frac{\text{side adjacent to the angle}}{\text{hypotenuse}}$

From the diagram 1 above

$\cos P = \frac{|RP|}{|PQ|}$

$\cos Q = \frac{|RQ|}{|PQ|}$

**Definition:** In a right triangle with acute angle $a$, the ratio of the leg adjacent to angle $a$ to the hypotenuse is called the *cosine* of angle $a$, abbreviated $\cos a$.

Notice that the value of $\cos a$, like that of $\sin a$, depends only on $a$ and not on the right triangle that includes $a$. Any two such triangles will be similar, and the ratio $\cos a$ will thus be the same in each.

**The Tangent of an Angle**

The abbreviation *tan* is usually used for tangent.

Tangent of an angle $= \frac{\text{side opposite to the angle}}{\text{side adjacent to the angle}}$

The simplest way to remember this is
Tangent of an angle = TOA

Where

T = Tangent of an angle

O = Opposite

A = Adjacent

Tangent of an angle = TOA = \[
\frac{\text{side opposite to the angle}}{\text{side adjacent to the angle}}\]

From the diagram 1 above

\[
\tan P = \frac{|RQ|}{|RP|}
\]

\[
\tan Q = \frac{|RP|}{|RQ|}
\]

You may notice that the tangent represents the gradient of a line (usually defined as the change in \(y\) divided by the change in \(x\). the word “tangent” is shortened to “tan” with \(\tan \theta\) denoting the tangent of the angle \(\theta\), again there is no multiplication involved.

Interestingly you can use the definitions sine and cosine above to show that

\[
\tan \theta = \frac{\sin \theta}{\cos \theta} \quad \tan \phi = \frac{\sin \phi}{\cos \phi}
\]

In other words:

**The tangent of an angle is the sine of that angle divided by its cosine**

**TRY QUESTIONS**

1. Use the figure below to answer the questions

   ![Diagram](image)

   a) Find \(\sin(\theta)\).

   b) Find \(\cos(\theta)\).

   c) Find \(\tan(\theta)\).

   d) Find \(\sin^2(\theta) + \cos^2(\theta)\).
2. Assume that $\sin(\alpha) = \frac{5}{13}$, where $\alpha$ is acute. (A)

a) Find $\cos(\alpha)$.

b) Find $\tan(\alpha)$.

c) Find $\sin^2(\alpha) + \cos^2(\alpha)$

Finding the value of a trigonometric ratio.

EXAMPLE 1

Given the $90^\circ$ triangle below, determine the angle $\theta$

Solution

Using $\Theta$ as our reference, we can see that side opposite was given, 6cm. Also, the hypotenuse was given, 10cm. To determine angle $\Theta$, we will use sine to help us, because it uses the side opposite and hypotenuse to determine angle $\Theta$

Sine of an angle = $\frac{\text{side opposite to the angle}}{\text{hypotenuse}}$

Sine of an angle = $\frac{6}{10} = 0.6$

Then convert to a degree or angle using inverse of sin

$\sin^{-1}(0.6) = 36.87^\circ$

$\tan^{-1}(0.75) = 36.87^\circ$

Check your Progress

Q.1 If $\sin \theta = \frac{2}{3}$, and the terminal side of the angle lies in the second quadrant, find the remaining trigonometric ratios of $\theta$
Q.2 If \( \sin \theta = \frac{3}{8} \), and the terminal side of the angle lies in the second quadrant, find the remaining trigonometric ratios.

Q.3 If \( \cos \theta = -\frac{\sqrt{3}}{2} \) and the terminal side of the angle lies in the third quadrant, find the remaining trigonometric ratios of \( \theta \).

Q.4 If \( \tan \theta = \frac{3}{4} \), and the terminal side of the angle lies in the third quadrant, find the remaining trigonometric ratios of \( \theta \).

**Suggested Answers**

Q.1 \[
\sin \theta = \frac{2}{3} \\
\cos \theta = -\frac{\sqrt{5}}{3} \\
\tan \theta = -\frac{2}{\sqrt{5}}
\]

Q.2 \[
\sin \theta = \frac{3}{8} \\
\cos \theta = -\frac{\sqrt{55}}{8} \\
\tan \theta = -\frac{3}{\sqrt{55}}
\]

Q.3 \[
\sin \theta = -\frac{1}{2} \\
\cos \theta = -\frac{\sqrt{3}}{2} \\
\tan \theta = \frac{1}{\sqrt{3}}
\]

Q.4 \[
\sin \theta = -\frac{3}{5} \\
\cos \theta = -\frac{4}{5} \\
\tan \theta = \frac{3}{4}
\]

**TRIGONOMETRY APPLICATION TO REAL WORLD**

Some problem-solving questions involves:

1. ‘Angle of Elevation’
2. ‘Angle of Depression’
ANGLE OF ELEVATION

Imagine a course mate is on the 3rd floor of the Faculty block. If you, standing on the ground floor looks up to find him, there is an angle formed between the ground level and the line of your sight.

Again, Mr. Abaidoo looked upwards at an orange hanging on top of his tree. There is an angle formed between the horizontal and the line of his sight in order for him to see the top of the tree. This angle is what is referred to as the Angle of Elevation.

The angle of Elevation of an object, B is from an observer at A, who is below the level of B, is the angle which AB makes with the horizontal.

For instance

The length of the shadow of the faculty block, 89m tall is 93cm. find the angle of elevation of the sun.

Solution

\[ \tan B = \frac{\text{side opposite to the angle}}{\text{side adjacent to the angle}} \]

\[ \tan B = \frac{AB}{AC} = \frac{89}{93} \]

\[ B = \tan^{-1}(0.957) \]

\[ = 43.74^\circ \]

Therefore, the angle of elevation is 44°

ANGLE OF DEPRESSION
In the same but opposite vein, if a course mate is on the 3\textsuperscript{rd} floor of the Faculty block of UEW and you standing on the ground floor, the course mate looks downwards to find you, there is an angle formed between the line of his sight and the ground level. Again, if Mr. Baidoo happened to be on top of the tree and looked downwards at an orange that has fallen on the ground. The angle formed between the line of his sight and horizontal ground in order for him to see the orange now becomes the \textit{Angle of Depression}.

In other words, if you look downwards at an object, the angle formed between the horizontal and your sight in order to see the object below is called the angle of depression.

The angle of depression of an object Q from an observer at R, who is above the level of Q, is the angle which RQ makes with the horizontal.

\begin{center}
\textbf{EXAMPLE 1}
\end{center}

An airplane is 1200m above the ground, its angle of elevation from point P on the ground is 30\textdegree. How far is the airplane from P by line of sight?

\textbf{Solution}

\begin{align*}
\sin 30^\circ &= \frac{\text{SOH}}{\text{hypotenuse}} \\
&= \sin 30^\circ = \frac{AB}{AP} \\
&= \sin 30^\circ = \frac{1200}{AP}
\end{align*}
\[ 0.5 = \frac{1200}{AP} \]

\[ AP = \frac{1200}{0.5} = 2400 \text{m} \]

Therefore, the distance between the airplane and P is 2400m

EXAMPLE 2

If the shadow of a pole 12m high is 2/3 its length, what is the angle of elevation of the sun, correct to the nearest degree?

Solution

Let \( \theta \) = Angle of elevation

\[ \tan \theta = \frac{\text{side opposite to the angle}}{\text{side adjacent to the angle}} \]

\[ \tan \theta = \frac{ON}{OM} = \frac{12}{8} \]

\[ \theta = \tan^{-1}(1.5) \]

\[ = 56.31^\circ \]

Therefore, the angle of elevation is 56\(^\circ\)

EXAMPLE 3

A man standing 40m from a vertical pole observes that the angle of elevation of the top of the pole is 18\(^\circ\). Assuming that his eye is 1.8m above the base of the pole, calculate to the nearest metre, the height of the pole.

Solution
Form the diagram, BD represent the Pole and OA represent the man. Hence,

\[
\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{side opposite to the angle}}{\text{side adjacent to the angle}}
\]

\[
\tan 18^\circ = \frac{BC}{AC} = \frac{h}{40}
\]

\[
h = 40 \times \tan 18
\]

\[
h = 40 \times 0.3249
\]

\[
h = 12.996 \text{m}
\]

therefore, the height of the pole BD

\[
= 12.996 + 1.8
\]

\[
= 14.796 \text{m}
\]

The height of the pole = 15m

EXAMPLE 4

A ladder leans against a wall. The end of the ladder touches the wall 12m from the ground. The foot of the ladder is 9m away from the foot of the wall. Find

i. The length of the ladder

ii. The angle that the ladder makes with the ground

Solution
Let the length of the ladder be \( x \)

Let the angle that the ladder makes with the ground be \( \theta \)

i. Applying Pythagoras theorem;

\[
|x|^2 = |12|^2 + |9|^2
\]

\[
|x|^2 = 144 + 81
\]

\[
|x|^2 = 225
\]

\[
|x| = \sqrt{225}
\]

\[x = 15\]

Therefore, the length of the ladder is 15m

ii. From the diagram,

\[
\tan \theta = \frac{\text{side opposite to the angle}}{\text{side adjacent to the angle}}
\]

\[
\tan \theta = \frac{12}{9} = \frac{4}{3}
\]

\[
\theta = \tan^{-1}(1.333)
\]

\[\theta = 53^\circ\]

Therefore, the angle that the ladder makes with the ground is 53°

---

Example 5

Two observers, P and Q 32m apart observe a kite (K) in the same vertical plane and from the same side of the kite. The angle of elevation of the kite from P and Q are 35° and 52° respectively. Find the height of the kite correct to two decimal places.

**Solution**
Let $|OQ| = x$ km

From triangle OPK

$\tan 32^\circ = \frac{OK}{OP}$

$0.7002 = \frac{OK}{32 + x}$

$OK = 0.7002 \times (32 + x) \ldots \ldots \ldots \ldots \ldots \ldots \ldots (1)$

From triangle OQK

$\tan 52^\circ = \frac{OK}{OQ}$

$1.2799 = \frac{OK}{x}$

$OK = 1.2799x \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (2)$

From equation 1 and 2

$1.2799x = 0.7002 \times (32 + x)$

$1.2799x = 22.4064 + 0.7002x$

$0.5797x = 22.4064$

$x = \frac{22.4064}{0.5797}$

$x = 38.652$

$OK = 1.2799x$

$= 1.2799 \times 38.652$

$= 49.471$ m

Therefore, the height of the kite from the ground is 49.47 m

Check Your Progress
1. The foot of a ladder is 2m from a wall and the top of the ladder is 6m above the ground. Calculate the length of the ladder.

2. A ladder leans against a vertical wall of height 15m. If the foot of the ladder is 8m away from the wall, find the length of it.

3. A ladder is 25m long. The foot of the ladder is 7m away from the base of the wall. How far up the wall is the top of the ladder?

4. A man standing 32m away from a tower observes that the angle of elevation of the top and bottom of a flagstaff standing on the tower at 62° and 60° respectively. Calculate the height of the flagstaff.

5. A pyramid in Egypt has a square base of side 700ft long and a height of the pyramid was 200ft. Find the angle of elevation.

6. The angle of the top of Simpa summer hut to the top of the Simpa block ‘A’ building is 46°40’ while the angle of depression to the bottom is 14°10’. If the summer hut is 28cm high, find the height of the Simpa block A building.

7. From the window, 30ft above the street, the angle of elevation to the top of my hostel is 50° and the angle of depression to the base of the hostel is 20°. Find the height of the hostel across the street.

Suggested Answers

5. 22°
6. 146°
7. 128ft

Summary

In sum, we have learnt that:

- When someone looks upwards at an orange hanging on top of his tree. There is an angle formed between the horizontal and the line of his sight in order for him to see the top of the tree.
- This angle is what is referred to as the Angle of Elevation.
- When someone at the top of a tree looks downwards at an orange that has fallen on the ground.
- The angle formed between line of his sight and horizontal ground in order for the person at the top of the tree to see the orange on the ground now becomes the Angle of Depression.
Chapter 3 **SECTION 4: ALGEBRA OF VECTOR AND ITS NOTATION**

Hello dear, you are now on unit 3 section 4 of this course. Before we proceed to this section to explore other aspects of vectors, let’s try to understand what the term” vector” in mathematics means. What do you think a vector is?

**Section objectives**

By the end of this section, you will be able to;

- Describe what a vector is
- State and briefly explain the main component of a vector
- Represent given vectors in algebraic forms
- Outline the various forms of vectors
- State whether a quantity is a vector or scalar quantity
- Calculate for the missing component of a vector

**Definition of vectors**

Otoo and Aboagye (2019) explained vector as” the translation or displacement of a certain distance in a certain direction.” Two key terminologies are emphasized in this definition. What are they? Obviously, they are Distance and Direction. This means that a vector describes any quantity that has magnitude and a direction.

A **vector** is a line segment. It has a starting point and an ending point. A vector has direction and length. It may be easiest to think of a vector as a line on a road map. There are usually several ways to describe the route. Look at the diagram below;

![Diagram](image)

**Components of Vectors**

Telling someone to get from the beginning to the end, you could say “Go 5 miles at an angle (in standard position) of 45 degrees.” This direct route is a vector. In this case, 5 is the **magnitude**, or length, of the vector, and 45° is the **amplitude**, or direction, of the vector.

Often, when we travel, we cannot go along the vector. Suppose you want to tell a neighbor how to go from your apartment to the library. In the diagram above, you may tell them to “Go east 3.5
miles then north 3.5 miles.” You are breaking the directions into **components**. We often break vectors into two components; the \( i \) component is the number of units in the x-axis direction and the \( j \) component is the number of units in the y-axis direction.

A **vector** (also called a **direction vector**) is just something that has both **magnitude** (length, or size) and **direction**. It is different from a regular number, since it really has two components to it. We see vectors represented by **arrows**, so we can remember that we need to get a length of a vector (the magnitude), as well as the direction (which way it’s pointing).

We use vectors in mathematics, engineering, and physics, since many times we need to know both the size (length) of something and which way it’s going. For Example, with an airplane, we can use a vector to measure the speed of the plane (the “size”) and the direction it’s flying.

**Scalar Quantity**

One key concept that relate to vector is Scalar. A **scalar** is a mathematical quantity with magnitude only (in physics, mass, pressure or speed are good examples).

**VECTOR REPRESENTATION NOTATION**

Let us remember that two-dimensional vectors can be represented in different ways. Among these ways are; To show that a quantity is a vector, it is printed in bold as \( \mathbf{m} \) or underline as \( \underline{m} \). Whenever the displacement is of \( x \) unit in the x-direction and \( y \)-unit in the y-direction, the vector is written as; \(( x, y )\)

a. **Geometrical Notation**

Here, we use an arrow to represent a vector. Its length is its **magnitude**, and its direction is indicated by the direction of the arrow.

![vector diagram](image)

The vector here can be written \( \mathbf{OQ} \) (bold print) or \( \overrightarrow{OQ} \) with an arrow above it. Its **magnitude** (or length) is written \(| \mathbf{OQ} |\) (absolute value symbols). An alternate notation is the use of two-unit vectors \( \mathbf{i} (1, 0) \) and \( \mathbf{j} (0, 1) \)
b. **Rectangular Notation** \((a, b)\)

A vector may be located in a rectangular coordinate system, as is illustrated here.

The rectangular coordinate notation for this vector is \(v (6, 3)\) or \(\vec{v} (6, 3)\). Note the use of **angle brackets** here.

**Amplitude** is the angle of the vector. We measure the angle as if the vector was the **terminal side** of an angle in **standard position**.

---

**Forms of vectors**

A vector can take one of the following forms;

1. **Distance bearing form**

Example: \((k, \theta)\)

2. **Component form**

Example: \(\begin{pmatrix} x \\ y \end{pmatrix}\)

3. **Cartesian form**

Example: \((xi + yj)\)

---

**Finding missing components of vectors**

When a vector is from a point say \(A (a, b)\) to another point \(B (a, b)\), then the vector is represented as \(\vec{AB}\).
Similarly, a vector from the origin, O to the points A and B is written as $\vec{OA}$ and $\vec{OB}$ with components 
\[
\begin{pmatrix} x \\ y \\ \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a \\ b \\ \end{pmatrix}
\]
respectively.

When the origin of the point A and B is say P instead of O, then

\[
\vec{AB} = \vec{PB} - \vec{PA}
\]

**EXAMPLE 1**

If the coordinate of the points P and Q from the origin O are (-3,5) and (7,2) respectively, find $\vec{PQ}$

**Solution**

The vector PQ should be calculated as the distance from the origin (O) hence;

\[
\vec{PQ} = \vec{OQ} - \vec{OP}
\]

\[
\vec{PQ} = \begin{pmatrix} 7 \\ 2 \\ \end{pmatrix} - \begin{pmatrix} -3 \\ 5 \\ \end{pmatrix}
\]

\[
= \begin{pmatrix} 7+3 \\ 2-5 \\ \end{pmatrix}
\]

\[
= \begin{pmatrix} 10 \\ -3 \\ \end{pmatrix}
\]

**EXAMPLE 2**

Given the vector $\vec{PQ} = \begin{pmatrix} -6 \\ 8 \\ \end{pmatrix}$ and $\vec{PR} = \begin{pmatrix} -2 \\ 4 \\ \end{pmatrix}$, Find $\vec{RQ}$

**Solution**

\[
\vec{RQ} = \vec{OQ} - \vec{OR}
\]

\[
\vec{RQ} = \vec{PQ} - \vec{PR} \quad \text{(since P is common)}
\]

\[
= \begin{pmatrix} -6 \\ 8 \\ \end{pmatrix} - \begin{pmatrix} -2 \\ 4 \\ \end{pmatrix}
\]

\[
= \begin{pmatrix} -6+2 \\ 8-4 \\ \end{pmatrix}
\]

\[
= \begin{pmatrix} -4 \\ 4 \\ \end{pmatrix}
\]

**Check Your Progress**
1. State whether each quantity described is a vector or scalar quantity.
   i. A hockey puck shot northwest at 60 miles per hour.
   ii. A tennis ball served at 110 miles per hour.
   iii. A parachutist falling straight down at 12 miles per hour.
2. Given the vector, $\overrightarrow{MN} = \begin{pmatrix} -10 \\ -2 \end{pmatrix}$ and $\overrightarrow{MQ} = \begin{pmatrix} 9 \\ 4 \end{pmatrix}$, find $\overrightarrow{NQ}$.
3. If the points A(-5, 2), B(3,4) and C(7, -6), find;
   (i) $\overrightarrow{AB}$  (ii) $\overrightarrow{BC}$
   (iii) $\overrightarrow{CA}$  (iv) $\overrightarrow{CB}$
   (V) $\overrightarrow{BA}$

Suggested answers
   1. i. Vector: northwest (direction) and 60 miles (distance)
      ii. Scalar: only distance (110 miles)
      iii. Vector: straight down (direction) and 12 miles (distance)

Summary
In this section, we have learnt that;

- A vector is a line segment. It has a starting point and an ending point.
- A vector describes any quantity that has magnitude and a direction.
- There are two major component of a vector; Distance and Direction.
- The distance of a vector is also called its magnitude
- We use vectors in mathematics, engineering, and physics, which usually involves the size of something (length) and which way it’s going
- One key concept that relate to vector is Scalar.
- A scalar is a mathematical quantity with magnitude only
- Examples of scalar quantities in physics are mass, pressure or speed.
- Two-dimensional vectors can be represented in different ways
- Among these ways are to show that a quantity is a vector, it is printed in bold or underlined as \( \mathbf{m} \) or \( \textbf{m} \) respectively.
- Also, whenever the displacement is of x unit in the x-direction and y-unit in the y-direction, the vector is written as; \( \begin{pmatrix} x \\ y \end{pmatrix} \)
- A vector can be represented using the geometric or rectangular notation.
- When a vector is from a point say A (a, b) to another point B (a, b), then the vector is represented as $\overrightarrow{AB}$.
• Also, a vector from the origin, O to the points A and B is written as $\overrightarrow{OA}$ and $\overrightarrow{OB}$ with components $(x, y)$ and $(a, b)$ respectively.

• When the origin of the point A and B is say P instead of O, then $\overrightarrow{AB} = \overrightarrow{PB} - \overrightarrow{PA}$. 
Chapter 4 SECTION 5: VECTOR OPERATIONS

My dear students, you have done very well so far and I hope you can recall all that we have learnt from section 4 of this unit. This is the fifth section of unit 3 and we want to explore the various operations with vectors. We shall look at addition, subtraction and scalar multiplications involving vectors.

Section objectives

By the end of this section, you will be able to find;

• The resultant vector of two or more vectors
• Difference of two or more given vectors
• Product of two or more scalar-vectors
• Solve problems involving addition, subtraction and multiplication of vector quantities.

• Addition

Addition of vectors can be expressed by a diagram. Placing the vectors end to end, the vector from the start of the first vector to the end of the second vector is the sum of the vectors. One way to think of this is that we start at the beginning of the first vector, travel along that vector to its end, and then travel from the start of the second vector to its end. An arrow constructed between the starting and ending points defines a new vector, which is the sum of the original vectors. Algebraically, this is equivalent to adding corresponding terms of the two vectors:

\[
\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_x \\ a_y \end{bmatrix} + \begin{bmatrix} b_x \\ b_y \end{bmatrix} = \begin{bmatrix} a_x + b_x \\ a_y + b_y \end{bmatrix}.
\]

Addition of vectors is also called resultant of vectors. Resultant is a single vector that gives the total effect of number of vectors. Resultant can be found by using a) Triangle law of vectors
b) Parallelogram law of vectors

c) Polygon law of vectors

Two vectors can be added either by triangle law or parallelogram law of vectors. Addition follows the parallelogram construction. Subtraction (a - b) is defined as the addition (a + (-b)). It is useful to remember that the vector a - b goes from b to a. The following results follow immediately from the above definition of vector addition:

(a) \( a + b = b + a \) (commutativity)
(b) \( (a + b) + c = a + (b + c) = a + b + c \) (associativity)
(c) \( a + 0 = 0 + a = a \), where the zero vector is \( 0 = [0, 0, 0] \).
(d) \( a + (-a) = 0 \)

**Subtraction**

Subtraction of vectors can be shown in a diagram form by placing the starting points of the two vectors together, and then constructing an arrow from the head of the second vector in the subtraction to the head of the first vector. The result of vector subtraction is called the *difference* of the two vectors. Algebraically, we subtract corresponding terms:

\[
\begin{align*}
\mathbf{a} - \mathbf{b} &= \begin{bmatrix} a_x \\ a_y \end{bmatrix} - \begin{bmatrix} b_x \\ b_y \end{bmatrix} \\
&= \begin{bmatrix} a_x - b_x \\ a_y - b_y \end{bmatrix}.
\end{align*}
\]

**Properties of vector addition and subtraction**

Several properties of vector addition are easily verified. For any vectors \( a, b, \) and \( c \) of the same size we have the following.

- Vector addition is *commutative*: \( a + b = b + a \).
- Vector addition is *associative*: \( (a + b) + c = a + (b + c) \). We can therefore write both as \( a + b + c \).
- \( a + 0 = 0 + a = a \). Adding the zero vector to a vector has no effect. (This is an Example where the size of the zero vector follows from the context: It must be the same as the size of \( a \).)
• $a - a = 0$. Subtracting a vector from itself yields the zero vector. (Here too the size of 0 is the size of $a$.)

Check your Progress

Given the vector $a = \begin{pmatrix} -7 \\ 0 \end{pmatrix}$, $b = \begin{pmatrix} 5 \\ 5 \end{pmatrix}$ and $c = \begin{pmatrix} 8 \\ -2 \end{pmatrix}$, find:

i. $a + b$
ii. $a - c$
iii. $b - c$
iv. $a - c + b$

Suggested Answers

i. $a + b = \begin{pmatrix} -7 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 0 + 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$

ii. $a - c = \begin{pmatrix} -7 \\ 0 \end{pmatrix} - \begin{pmatrix} 8 \\ -2 \end{pmatrix} = \begin{pmatrix} -7 - 8 \\ 0 + 2 \end{pmatrix} = \begin{pmatrix} -15 \\ 2 \end{pmatrix}$

iii. $b - c = \begin{pmatrix} 5 \\ 5 \end{pmatrix} - \begin{pmatrix} 8 \\ -2 \end{pmatrix} = \begin{pmatrix} 5 - 8 \\ 5 + 2 \end{pmatrix} = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$

iv. $a - c + b = \begin{pmatrix} -7 \\ 0 \end{pmatrix} - \begin{pmatrix} 8 \\ -2 \end{pmatrix} + \begin{pmatrix} 5 \\ 5 \end{pmatrix} = \begin{pmatrix} -7 - 8 + 5 \\ 0 + 2 + 5 \end{pmatrix} = \begin{pmatrix} -10 \\ 7 \end{pmatrix}$

Scalar-Vector Multiplication

Another operation is scalar multiplication or scalar-vector multiplication, in which a vector is multiplied by a scalar (i.e., number), which is done by multiplying every element of the vector by the scalar. Scalar multiplication is denoted by juxtaposition, typically with the scalar on the left, as in

$$(-2) \begin{pmatrix} \frac{1}{3} \\ -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} -2\left(\frac{1}{3}\right) \\ -2\left(-\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}$$

Where
$\vec{AB} = \begin{pmatrix} x \\ y \end{pmatrix}$ then $k \vec{AB} = \begin{pmatrix} kx \\ ky \end{pmatrix}$ where $k$ is a **scalar or number** which can be negative or positive whole number or fraction. When $k$ is positive, it implies the vectors are parallel and in the same direction. When $k$ is negative, it means the vectors are parallel but in opposite directions. The length of the new vector is $k$ times the length of the original vector.

NB: To find the scalar multiplication of a vector, multiply each component of the vector by the scalar.

**Example 1**

If $\vec{AB} = \begin{pmatrix} -1 \\ 5 \end{pmatrix}$ then $2 \vec{AB} = \begin{pmatrix} \text{?????} \end{pmatrix}$ then $3 \vec{AB} = \begin{pmatrix} \text{?????} \end{pmatrix}$

**Solution**

$2 \vec{AB} = \begin{pmatrix} 2 \times (-1) \\ 2 \times 5 \end{pmatrix} = \begin{pmatrix} -2 \\ 10 \end{pmatrix}$

and

$3 \vec{AB} = \begin{pmatrix} 3 \times (-1) \\ 3 \times 5 \end{pmatrix} = \begin{pmatrix} -3 \\ 15 \end{pmatrix}$

**Check Your Progress**

1. If $a = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$, $b = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$ and $c = \begin{pmatrix} -2 \\ -6 \end{pmatrix}$, then Find
   i. $2b$
   ii. $a - 2b + c$
   iii. $2(a + b)$

2. If $\vec{r} = \begin{pmatrix} 3 \\ 10 \end{pmatrix}$ and $\vec{s} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$, calculate $6(\vec{r} + 2\vec{s})$

**Suggested answers**

1. i. $\begin{pmatrix} 8 \\ -10 \end{pmatrix}$ ii. $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ iii. $\begin{pmatrix} -14 \\ -8 \end{pmatrix}$

2. $\begin{pmatrix} -6 \\ 120 \end{pmatrix}$

**Summary**

In this section, we learnt that;

- Addition of vectors is also called resultant of vectors.
- Resultant is a single vector that gives the total effect of number of vectors.
- The result of vector subtraction is called the *difference* of the two vectors.
- Vector addition is *commutative* when $a + b = b + a$. 

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• Vector addition is associative when \((a + b) + c = a + (b + c)\). We can therefore write both as \(a + b + c\).
• Adding the zero vector to a vector has no effect, \(a + 0 = 0 + a = a\).
• Subtracting a vector from itself yields the zero vector, \(a - a = 0\).
• To find the scalar multiplication of a vector, multiply each component of the vector by the scalar.
Chapter 5 SECTION 6: MAGNITUDE AND DIRECTION OF A VECTOR

Dear student, we welcome you to the last section of unit 3, which is section 6. The section will consist of two parts. The first part of this section will look at magnitude of vectors while the last part will talk about its direction. We encourage you to use your prior learning of coordinate geometry of determining the magnitude of a line to help you understand the magnitude of a vector better. Do well to also link and apply our previous lesson on trigonometric ratio to the subsequent lesson on direction of a vector.

In this first part of the section, we seek for you to gain a deeper understanding of magnitude as a component of vector. Do you remember what we learnt earlier about the component of vectors? Obviously, you recall that a vector quantity has magnitude and direction. Displacement, velocity, momentum, force, and acceleration are all vector quantities. For Example, in Greek mathematics textbooks, a vector is determined by three components: magnitude, path, sense, whereas, in Greek physics textbooks, a vector quantity is determined by two components: magnitude and direction (including the concepts of path and sense)

Section objectives

By the end of this section, you will be able to;

- Find the length or magnitude of a vector using the distance formula
- Calculate the unknown sides of a given triangle
- Express given points as vectors
- Prove a type of triangle using given sides of the triangle
- Identify the four cardinal directions used in bearings
- Describe the main types of bearings
- Calculate the back bearings of a given distance

Maginudes of Vectors

The length of a vector is its magnitude. The symbol for magnitude of a vector can be written $|v|$ or like an absolute value, $|v|$, or just written “magnitude”. If you are given the components of a vector, use the distance formula to find the magnitude.

To calculate the magnitude (length) of a vector, we may use some plane geometry. First, sketch your vector on the coordinate plane. Because the angle between the $x$- and $y$-axes is $90^\circ$, it should be straightforward to draw a right triangle with the arrow as hypotenuse and sides parallel to the axes. Then the length of the hypotenuse may be calculated from the lengths of the sides using the Pythagorean theorem $c^2=\sqrt{a^2+b^2}$ as in the Example below.
**Example**

What is the magnitude of vector $\mathbf{A}$ drawn in the diagram below?

![Diagram with vector A](image)

**Solution**

If you zoom in on the arrow, the blown-up picture looks like: where the vector arrow is the hypotenuse of a right triangle that has a horizontal side with length of 4 units and vertical side of length 1 unit.

Plugging this into the Pythagorean theorem:

$$c^2 = a^2 + b^2$$

$$\sqrt{4^2 + 1^2}$$

$$\sqrt{16 + 1}$$

$$\sqrt{17}$$

If you wish, you could also plug the coordinates into the distance formula,

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

As this formula is derived by putting generic coordinate points into the Pythagorean theorem, the calculation is essentially identical to the geometric method used in the Example. In the Example above, if you plug the points $(x_1, y_1) = (5, 1)$ and $(x_2, y_2) = (1, 2)$ into the distance formula, you will calculate the same result as $\sqrt{17}$

Since the magnitude of a vector is like a length measurement, the magnitude is always positive.
Example 1

if $\overrightarrow{PQ} = (-6, 8)$, find $|\overrightarrow{PQ}|$

Solution

$|\overrightarrow{PQ}|$ is the length of $\overrightarrow{PQ}$

$|\overrightarrow{PQ}| = \sqrt{x^2 + y^2}$

$= \sqrt{(-6)^2 + 8^2}$

$= \sqrt{36 + 64}$

$= \sqrt{100}$

$= 10$ unit

Expressing Two Given Points as Vectors

If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two given points, then the vector joining $A$ and $B$ is given by

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = a - b$

$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} - \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} x_2 - x_1 \\ y_2 - y_1 \end{pmatrix}$

Similarly, given the points $X$ and $Y$, the vector $\overrightarrow{XY} = \overrightarrow{OY} - \overrightarrow{OX} = y - x$

Also,

$\overrightarrow{CD} = d - c$

NB: Note the order of the subtraction thus, the vector $\overrightarrow{AB} = \text{position vector of B} - \text{position vector of A}$

Check Your Progress

Example 1

Find $\overrightarrow{AB}$ and $\overrightarrow{BA}$ if $A(4, 8)$ and $B(-1, 5)$
Example 2
A triangle ABC has vertices A (-2, -4), B (10,1) and C (3,8). Find the length of the sides AB and AC. Show that the triangle is isosceles.

Example 3
A triangle XYZ has vertices X (5,5), Y (6,2) and Z (4,8). Find whether triangle XYZ is an equilateral triangle.

Suggested Answers
1. \[ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} - \begin{pmatrix} -5 \\ -3 \end{pmatrix} = \begin{pmatrix} 4 \\ 7 \end{pmatrix} \]
\[ \overrightarrow{BA} = \overrightarrow{OA} - \overrightarrow{OB} = \begin{pmatrix} 4 \\ -1 \end{pmatrix} - \begin{pmatrix} -5 \\ 5 \end{pmatrix} = \begin{pmatrix} 9 \\ -6 \end{pmatrix} \]

2. Length of AB = \[ \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(10 - (-2))^2 + (1 - (-4))^2} = \sqrt{12^2 + 5^2} = \sqrt{144 + 25} = \sqrt{169} = 13 \text{ units} \]

Length of AC = \[ \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(3 - (-2))^2 + (8 - (-4))^2} = \sqrt{5^2 + 12^2} = \sqrt{25 + 144} = \sqrt{169} = 13 \text{ units} \]

Therefore, since the two sides of the triangle are of the same length=13 units, the triangle is isosceles because two of its sides are equal.

3. For XYZ to be an equilateral triangle, each of the length of its sides should be of the same length. Find the length of AB, BC and AC to be sure they are of same length.

Summary
In a nutshell, we have learnt that;

- If A \((x_1, y_1)\) and B \((x_2, y_2)\) are two given points, then the vector joining A and B is given by \[ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = a - b \]
- The magnitude of a vector is its length.
- The symbol for magnitude of a vector can be written \( |v| \), or just written “magnitude”
- We use the distance formula to find the magnitude of a vector
- To find the length of the hypotenuse, you calculate from the lengths of the sides using the Pythagorean theorem \( c^2 = a^2 + b^2 \)
- Since the magnitude of a vector is like a length measurement, the magnitude is always positive.
DIRECTION OF A VECTOR (BEARINGS)

Dear student, we welcome you once more to the second part of section 6 of unit 3. In this part of the section, we aim at helping you to gain a deeper understanding of another major component of vectors. Do you remember what we learnt earlier about the component of vectors? Obviously, you recall that a vector quantity has magnitude and direction. In the first part of this unit, we learn a lot about the magnitude of a vector. We now want to throw a spotlight on the second component of a vector; Direction. Imagine a motorist riding from left of a point P to another point Q of 300m on an angle of 120° observed a beer bar (R) 150m away on angle of 050°. How far west do you think is P from the beer bar? How far north is also the point R from Q.

The ideas from these situations in our introduction bring to mind the idea of bearings. What then is a bearing? Let us look at it. A bearing is an angle measurement made with reference to the north and measured in clockwise direction usually expressed as a three-digit number of figures. This means that its measurement starts from the north pole.

A bearing gives direction in terms of an angle.

The four cardinal directions are

i. North (000°)
ii. South (180°)
iii. East (090°)
iv. West (270°)

The directions NE, NW, SW, SE are frequently used for compass bearings.
Kinds of bearing

There are two kinds of bearing

i. Compass bearings
ii. Three-figure bearings

A. Compass Bearing

Compass bearings are based on the four main cardinal points; North, South, East and West. It is an angle measured from the north or south. The directions half-way between the cardinals are marked in the figure below;

For Example

**N 50° E** means from N measure 50° towards E where N and E are North and East respectively.

**S 40° W** from S measure 40° towards W.

**N 20° E** is pronounced as 20° north east.

The compass bearing of **N 32° E** means an angle of 32° measured from the north towards east and the 3-digit bearing is \( (90° - 32°) = 58° \)

Again, a compass bearing of **N50°W** means an angle of 50° measured from the north towards west and the 3-digit bearing is \( (360° - 50°) = 310° \)
Also, a compass bearing of S 15° E means an angle of 15° measured from the south towards east and the 3-digit bearing is (180° - 15°) = 165°.

Figure 1

Trial Question

Explain the compass bearing of S75°W. Find the 3-digit bearing.

B. Three-Figure Bearings

When we talk of bearings nowadays, we mean “three-figure bearings”. A three-figure bearing is always:

- An angle measured in degrees.
- Measured as angles in the clockwise direction from the geographic north.
- An angle given in three figure or digits from 000° to 360°
- We use extra zeros to make the number up to three digits if you need to. For Example, 9° gives a bearing 009° and 99° gives a bearing of 099°

Example

Given a vector in this form, (35km, 055°) means that the magnitude (length) of that vector is 35 km while its direction is 55°.

Typical 3-digit bearings

Some typical examples of three-digit bearings are shown below:
Types of Bearing

Back Bearing

Consider heading out to some place and have to return along the same line of travel. This is where the idea of back bearing becomes so useful.

Back bearing as the name indicates, is the reverse of a given bearing. For Example, the back bearing of the bearing P from Q is the bearing Q from P. In general, if the bearing of P from Q is \( \theta \), then the bearing of Q from P is

- a) \( \theta + 180^\circ \), if \( \theta \) is less than \( 180^\circ \)
- b) \( \theta - 180^\circ \), if \( \theta \) is more than or equal to \( 180^\circ \)

Calculating Back Bearings

There are two basic formulas for calculating a back bearing.

1. **When the direction of travel bearing is Less than 180°.** If for Example your bearing is 45, then your back bearing works out to 225°.

   Back bearing = (180° + Direction of travel bearing)

   \[
   \begin{align*}
   BB & = 180^\circ + B \\
   BB & = 180^\circ + 45^\circ \\
   & = 225^\circ
   \end{align*}
   \]
2. **When the direction of travel bearing is Greater than 180°.** If for Example, your bearing is 225°, which is greater than 180°, then your back bearing works out to be 45°

Back bearing = (Direction of travel bearing - 180°)

\[
BB = B - 180°
\]

\[
BB = 225° - 180° = 45°
\]

**Note**

1. \(\theta + 180°\), if \(\theta\) is less than 180°
2. \(\theta - 180°\), if \(\theta\) is more than or equal to 180°

**Example 1**

The bearing of Y from X is 320°. Find the bearing of X from Y.
Solution
The bearing of X from Y = \(320^\circ - 180^\circ = 140^\circ\)

**Example 2**
The bearing of a point B from a point C is \(062^\circ\). What is the bearing of C from B?

**Solution**
The bearing of C from B = \(62^\circ + 180^\circ = 242^\circ\)

**Example 3**
A ship is on a bearing of S65\(^\circ\)W from a harbor. What is the bearing of the harbor from the ship?

**Solution**
The back bearing in cardinal S65\(^\circ\)W is the same as N65\(^\circ\)E
So, the back bearing of the harbor from the ship is N65\(^\circ\)E

**Example 4**
If P is equidistant from Q and R. The bearing of Q from P is \(80^\circ\) and the bearing of R from P is \(130^\circ\). What is the bearing of R from Q?

**Solution**
From the question \(|PQ| = |PR|\) since P is equidistant from Q and R. The bearing of R from Q is measured from the north pole of Q in the clockwise direction to the line RQ

From \(\angle RPQ + \angle PQR + \angle PRQ = 180^\circ\)

\[50^\circ + 2\angle PQR = 180^\circ\]

\[2\angle PQR = 180^\circ - 50^\circ\]
2 \angle PQR = 130

\angle PQR = \frac{130}{2}

\angle PQR = 65^\circ

The bearing of R from Q = 270^\circ - (10^\circ+65^\circ)

= 195^\circ

**Example 5**

If \(\vec{PQ} = (5\text{km}, 240^\circ)\). Find \(3\vec{PQ}\)

**Solution**

\(3\vec{PQ} = (3 \times 5\text{km}, 240^\circ)\)

= (15\text{km}, 240^\circ)

---

**Let us explore more on bearings**

As stated earlier, the bearing of any vector is the direction which is calculated from the north through clockwise direction and expressed in three digits.

The component of a vector can be located in any of the four quadrant and its direction measured from the north. To calculate the bearing, trigonometric ratios are used.

**Example**

C (5,3) and D (7,5) are points in the same plane. Find the magnitude and bearing of D from C.

**Solution**

The bearing of D from C is the vector \(\vec{CD}\)

\(\vec{CD} = \vec{OD} - \vec{OC}\)

= \((7) - \left(\begin{array}{c}5 \\ 3 \end{array}\right)\)

= \left(\begin{array}{c}2 \\ 0 \end{array}\right)

The magnitude or length of \(\vec{CD}\)
\[ |\overrightarrow{CD}| = \sqrt{(2)^2 + (2)^2} \]
\[ = \sqrt{4 + 4} \]
\[ = \sqrt{8} \]
\[ = 2\sqrt{2} \]

\[
\tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{2}{2} = \tan^{-1}(1) = 45^\circ
\]

Check Your Progress

1. If \(\overrightarrow{AB} = (10 \text{km}, 060^\circ)\), find \(\overrightarrow{BA}\)

2. Three towns P, Q and R are such that P is 3km on a bearing of 090° from Q. R is 9km due south of P. find the bearing of Q from R.

3. A jet fighter flew from its base to the enemy’s camp, a distance of 510km on a bearing of 060°. It was then forced to fly a distance of 450km on a bearing of 150° to a fuel station to refill. Calculate
   i. The distance between the fuel station and the base (2 decimal place)
   ii. The bearing of the base from the fuel station to the nearest whole number.

4. If the bearing of P from Q is 145°. Find the bearing of Q from P.

5. Three towns P, Q and R in the same plane are such that \(\overrightarrow{PQ} = (32 \text{km}, 222^\circ)\) and \(\overrightarrow{QR} = (56 \text{km}, 312^\circ)\). Calculate
   a) \(|\overrightarrow{PR}|\)
   b) The bearing of Q from R
   c) A town M is on \(\overrightarrow{PR}\) such that \(|\overrightarrow{PM}| : |\overrightarrow{MR}| = 2:3\), find \(|MQ|\)
6. A (-3,5) and B (-1,2) are two points. Express AB in the form (k, \( \alpha \)) where k is the magnitude and \( \alpha \) is the bearing.

d) The points M, N and R are three villages in the same plane of a district. \( \overrightarrow{MN} \) = (y km, 042\(^\circ\)), \( \overrightarrow{RN} \) = (22 km, 315\(^\circ\)) and \( \overrightarrow{MR} \) = 32 km.

a. Find y
b. T is a point on \( \overrightarrow{NR} \) such that \( \angle MTN = 30^\circ \) find \( \overrightarrow{MT} \)

Suggested Answers

1. \( \overrightarrow{BA} \) = (10 km, 180\(^\circ\) + 60\(^\circ\)) = (10 km, 240\(^\circ\))

2. Let \( \theta \) = angle of QRP

From \( \triangle PQR \), \( \tan \theta = \frac{\text{opposite}}{\text{adjacent}} = \frac{3}{9} = \tan^{-1}(0.333) = 18.42^\circ \).

Therefore, the bearing of Q from R = 360\(^\circ\) - 18.42\(^\circ\) = 342\(^\circ\)

3. See figure for question 4 below;

i. From \( \triangle BCS \)

\[ |BS| = \sqrt{(510)^2 + (450)^2} \]
\[
\begin{align*}
&= \sqrt{260100 + 202500} \\
&= \sqrt{462600} \\
&= 680.147
\end{align*}
\]

The distance between the base and the fuel station is 680.147km

ii. From \(\triangle BCS\), let \(\beta = \angle BSC\)

\[
\tan \beta = \frac{|BC|}{|BS|} = \frac{510}{450}
\]

\[
\beta = \tan^{-1} \left( \frac{510}{450} \right)
\]

\[
= 48.58^\circ
\]

Bearing of the base from the fuel station

\[
= 360^\circ - (48.58^\circ + 30^\circ)
\]

\[
= 360^\circ - 78.58^\circ
\]

\[
= 281.42^\circ = 281^\circ
\]

Summary

Recapping our lesson in this section, we learnt that:

- A bearing gives direction in terms of an angle.
- The bearing of any vector can be said to be the direction which is calculated from the north through clockwise direction.
- The four basic cardinal direction of a bearing are the North (000°), South (180°), East (090°) and West (270°).
- Bearings are expressed in three digits.
- A bearing can either be a compass bearings, three-figure bearings or back bearing.
- Back bearing is the reverse of a given bearing.
- When the direction of travel bearing is less than 180°, then your back bearing is worked out by adding it to 180°.
- When the direction of travel bearing is greater than 180°, then your back bearing is calculated by subtracting 180° from it.
UNIT 4

MENSURATION

Introduction

Dear student, you are heartily commended for journeying with us through the beginning units to the unit 4 of this course. It is our hope that you will be able to synchronize all that you have learnt in the first three units to help you gain conceptual grounds in this unit. In this fourth unit, you will be introduced to mensuration and it is a promise that you will come to love this unit also. The unit reintroduces the fundamentals of geometry; the three dimensions, and the origins of length, area and volume calculations. Concepts such as measurement of length (arc length, radius, diameter, chord), teaching area of a sector, area of segments volume of cone, cylinder as well as application of mensuration to real life problems will be brought up. It will finally reinforce the mathematical vocabulary related to the concept of mensuration.

Unit Content

This unit is about:

Section 1: Concept of Mensuration
Section 2: Parts of a Circle
Section 3: Measurement of Length (arc length, radius, diameter and a chord)
Section 4: Area of a Sector and Segment
Section 5: Volume of Cone and Cylinder
Section 6: Mathematical Vocabularies in Mensuration.
Chapter 6 SECTION 1: MENSURATION

Introduction

Dear student, welcome to the first section of unit 4 of this course. In this unit, we will take a walk to the world of mensuration to explore what the term is and the major component of mensuration.

Section objectives

By the end of this section, you will be able to;

- Explain briefly what mensuration is.
- Identify the major concepts in mensuration
- Describe the types of mensuration that are taught in schools

Mensuration

Dear student, you must have noted that since the beginning of civilization, man has explored the question of "how much"? Man started measuring land, water or milk etc. This was the origin of mensuration which later became the basis of architecture and engineering. Mensuration is a branch of mathematics which deals with lengths of lines, area of surfaces and volume of solids. A student studying mensuration should be conversant with metric system of units and their conversion. The concepts of area and volume should be known to the student. This brings us to another important unit of the course. In this unit, we shall discuss about mensuration, its teaching, learning and application. You have studied about perimeters and areas of plane figures like rectangles, squares, triangles, trapeziums, circles, kites etc. These are called plane figures because each of them lies in a plane. However, most of the objects that we come across in daily life do not lie in a plane. Some of these objects are bricks, balls, ice cream cones, drums, and so on. These are called solid objects or three-dimensional objects. The figures representing these solids are called three dimensional or solid figures. Some common solid figures are cuboids, cubes, cylinders, cones and spheres. Many students seem to get confused by whether they should use m², cm³ or whatever. They also sometimes confuse about the formulae for volume/area/length. This discussion is designed to review and reinforce the difference between these concepts.

Mensuration is the aspect of geometry that emphasize measurement of length of lines, perimeter, area, surface area and volume of both solid and plane shapes. The idea of geometry is useful to us as student so we can apply this in our real life of today.

Mensuration is a therefore a branch of mathematics which deals with the lengths of lines, areas of surfaces and volumes of solids.

Types of Mensuration
The two basic types of mensuration are:

1. Plane Mensuration
   It deals with the sides, perimeters and areas of plane figures of different shapes.

2. Solid Mensuration
   It deals with the areas and volumes of solid objects

Check Your Progress

1. How would you explain the term, mensuration to a friend?
2. Which type of mensuration would you recommend for learners to be introduced to first?
3. State any ten (10) key concepts that are taught in mensuration

Suggested Answers

1. Mensuration is a branch of mathematics which deals with the lengths of lines, areas of surfaces and volumes of solids. (Search online for other definitions)
2. We think solid mensuration should be fundamental since that is what is real to learners and is used in their daily activities
3. lines, length, areas, surfaces, volumes, cones, circles, triangles, sphere, cubes, cylinder, cuboid etc.

Summary

We have studied in this section that;

- Mensuration is an aspect of geometry.
- Mensuration emphasizes measurement of length of lines, perimeter, area, surface area and volume of both solid and plane shapes.
- Mensuration is also a branch of mathematics which deals with the lengths of lines, areas of surfaces and volumes of solids.
- The major types of mensuration are solid mensuration and plane mensuration.
- The major concepts in mensuration includes lines, length, areas, surfaces, volumes, cones, circles, triangles, sphere, cubes, cylinder, cuboid
Chapter 7 SECTION 2: A CIRCLE

Introduction

Dear students, you are now in section 2 of this unit. Your background knowledge of circles will be very helpful in this section. In this section, we shall talk extensively about circles and learn some basic terminologies that are associated with circle in mensuration. Stay with us as we go through these concepts together because of its applicability in the learning of the subsequent sections of this unit.

Section objectives

By the end of this section, you will be able to;

- Identify the major parts of any circle using the right terminology
- Apply circle terminologies in the next section to solve mensuration problems

A CIRCLE is the locus of a set of points which are equidistant (equal distance) from a fixed point.

![Diagram of a circle showing radius](image)

The circular part of a circle is called the CIRCUMFERENCE. Thus, it is the perimeter of the circle. See fig A

Any part of the circumference of a circle is called an ARC. Thus, the distance along the circumference is the arc. See fig

![Diagram of a circle showing an arc](image)

The fixed point is known as the CENTRE. In the figure above, the centre is the dot and usually termed as the origin (o)
The distance from the centre to any part or point of the circumference is called the **RADIUS**. Thus, a line joining the centre of a circle to any of the points on the circle is known as the radius. See fig A and B

A line that divides the circle into two equal parts is called a **DIAMETER**. A diameter is formed from two radius. Each of the radius is represented below as (r)

![Diagram of radius and diameter](image)

A straight line joining two points on the circumference is called a **CHORD**. Thus, the chord AB is the **Diameter**. The line CD forms a **chord** which is not a diameter. Can you explain why?

![Diagram of chord and segment](image)

A chord divides the circle into two parts called the **SEGMENT**. A segment is an area bounded by a chord and intercepted by an arc.
The area bounded by two radii of the circle is called a **sector** of the circle. The unshaded region in the figure above is a sector of the circle. See the two figures below;

The shaded part of the figure below is called a **minor sector** of the circle. See fig below;

The shaded part in the circle below is called the **major sector** of the circle.
When a circle is divided into four equal parts, one quarter of the area of the circle is called a QUADRANT. The shaded part of the figure below forms one of the four quadrant of a circle.

A SEMI-CIRCLE is half the area of the circle. Thus, when the circle is divided into two equal parts, each part becomes a semi-circle. See fig G

The names of the principal parts of a circle are shown below;

Check your progress

1. Define each of the following terminologies that are used in mensuration
   a) Circle
   b) Circumference
   c) Arc
   d) Origin of a circle
e) Radius
f) Diameter
g) Chord
h) Segment
i) Minor Segment
j) Major Segment
k) Sector
l) Quadrant
m) Semi-circle

2. Support your definitions with pencil sketches of each term

**Suggested answer**

1. Search online for other definitions of the terms.
2. Draw to show each term. Be accurate in your drawing

**Summary**

In summary, we have learnt that a circle has many key parts that are used in mensuration. Among these key parts of a circle are the Circumference, Arc, Origin of a circle, Radius, Diameter, Chord, Segment, Minor Segment, Major Segment, Sector, Quadrant and a Semi-circle. It is our hope that you will always remember these concepts to be able to apply them in our next section.
Chapter 8 SECTION 3: MEASUREMENT OF LENGTH

Introduction

Congratulation, my dear student for painstakingly going through the first two section of the unit 4 with us. Welcome to section 3 of unit 4 and in this section, we shall explore various concepts of length in mensuration. These concepts include learning about the length of arc, perimeter, circumference, and sector. It is our hope that your prior knowledge of circle theorem and parts of a circle will be used to aid the effective learning of measurement of length in this section.

Section objectives

By the end of this section, you will be able to find;

- Circumference of a given circle
- The length of a given arc
- Perimeter of a given sector

Chapter 9 A. length of an Arc

The length of arc of a circle is the measure of the part of a circumference.

The length of arc of a sector is the length of the portion on the circumference of the circle intercepted between the bounded radii.

In the figure, the length of arc AB of the sector AOB is l. if $\theta$ is the angle of the sector ad r is the radius of the circle from which the sector is, then the length of the arc of the sector will be the product of the ratio of the angle subtended by the arc to the total angle of the circle and the circumference of the circle.

Hence, the length of arc is given by

$$\frac{\theta}{360} \times \text{circumference of a circle}$$
\[
\frac{\theta}{360} \times 2\pi r
\]

Example 1

Find the length of an arc that subtend an angle of 80° at the centre of a circle with radius 7m

Solution

the length of arc \(= \frac{\theta}{360} \times 2\pi r\)

where \(\theta=85^\circ, r=7m, \pi = 3.142\)

\[
= \frac{80}{360} \times 2(3.142 \times 7)
\]

\[
= \frac{80}{360} \times 2(21.994)
\]

\[
= \frac{2}{9} \times 43.988
\]

\[
= 9.78m
\]

Hence, the length of the arc of that circle is 9.78m

Check Your Progress

1. The end of an arc subtends an angle of 120° at the centre of a circle of radius 21.7cm. Find the length of its arc.
2. The length of arc of a circle with radius 7.5cm is 18.33cm. Calculate the angle subtended at the centre by the arc.
3. An arc subtends an angle of 36° on the circumference of a circle whose diameter is 9.8cm. Find the perimeter of the major arc of the circle.

Suggested Answers

1. 45.5
2. 140°
Chapter 10 **B. Perimeter / Circumference of A Circle**

The circumference of a circle is the perimeter of that circle. Thus, it is the total distance around the circle. The circumference of a circle with radius \( r \) is given by

\[
C = 2\pi r
\]

Where \( \pi = 3.142 \text{ or } \frac{22}{7} \) and \( r = \text{radius} \)

However, 2 of the radius = diameter

\( = 2r = d \)

Substituting this into the formulae

\( = C = \pi d \text{ where } d = \text{diameter or } 2r \)

In conclusion, the circumference of a circle with radius \( r \) or diameter \( d \) is given by;

\[
C = 2\pi r \text{ or } C = \pi d
\]

**Example 1**

A circle has radius 7cm. Find its circumference.

Solution

\[
C = 2\pi r
\]

Where \( \pi = 3.142 \text{ or } \frac{22}{7} \) and \( r = \text{radius} \)

\[
C = 2 \times 3.142 \times 7
\]

\[
= 14 \times 3.142
\]

\[
= 43.988 \text{cm}
\]

**Example 2**

Find the perimeter of a circle whose radius is 3.2m.

Solution

Perimeter of that circle is the circumference of a circle
C = 2\pi r

Where \( \pi = 3.142 \text{ or } \frac{22}{7} \) and \( r = \text{radius} \)

\[
C = 2 \times 3.142 \times 3.2 \\
= 6.4 \times 3.142 \\
= 20.1 \text{m}
\]

**Check Your Progress**

Find the circumference of a circle with radius 8cm. (Take \( \pi = \frac{22}{7} \))

**Suggested answer**

Circumference = 2\pi r

Where \( \pi = 3.142 \text{ or } \frac{22}{7} \) and \( r = \text{radius} \)

\[
C = 2 \times 3.142 \times 8 \\
= 16 \times 3.142 \\
= 50.272 \text{cm}
\]

**Chapter 11 Perimeter of A Sector**

The perimeter of a sector is the sum of all the length of the boundaries of the sector. The perimeter of a sector whose length is \( L \) in a circle with a radius \( r \) is given by

\[
P = \text{length of the arc } + 2 \text{ of its radius}
\]

\[
P = L + 2r
\]

Where \( P = \text{perimeter, } L = \text{length of arc and } r = \text{radius} \)

**EXAMPLE**

If the radius of a sector is 21cm and its sector angle is 65\(^\circ\). Calculate the

a) Length of its arc  
b) Perimeter of the sector

140
SOLUTION

a) The length of arc is given by

\[ L = \frac{65}{360} \times 2\pi r \]

\[ L = \frac{65}{360} \times 2(3.142 \times 21) \]

\[ L = \frac{65}{360} \times 2(65.982) \]

\[ = \frac{13}{72} \times 2(65.982) \]

\[ L = 23.8 \text{cm} \]

b) Perimeter of the sector

\[ P = \text{length of the arc} + 2 \text{of its radius} \]

\[ P = L + 2r \]

Where \( P = \) perimeter, \( l = \) length of arc and \( r = \) radius

But \( L = 23.8 \text{cm} \) and \( r = 21 \)

\[ P = L + 2r \]

\[ P = 23.8 + 2(21) \]

\[ = 23.8 + 42 \]

\[ = 65.8 \text{cm} \]

Check Your Progress

1. Find the radius of a circle if the length of arc of the sector is 66cm and the sector angle is 60°.
2. Find the sector angle if the length of its arc is 22m and its radius is 7m.
3. A Seesaw swings through an angle of 85° with an arc length of 22cm. find the length of the seesaw rope.

Suggested answers

1. 63cm
2. 180°
Summary

In brief, we have learnt from the section that;

- The length of arc of a circle is the measure of the part of a circumference.
- The length of arc is given as $\frac{\theta}{360} \times$ circumference of a circle $= \frac{\theta}{360} \times 2\pi r$
- The circumference of a circle is the perimeter of that circle. Thus, it is the total distance around the circle.
- The circumference of a circle with radius (r) is given by $C=2\pi r$ Where $\pi = 3.142 \ or \ \frac{22}{7}$ and r = radius
- 2 of the radius of a circle makes a diameter hence the circumference of a circle is the same as $\pi d$
- The perimeter of a sector is the sum of all the length of the boundaries of the sector.
- The perimeter of a sector whose length is l in a circle with a radius r is given by $P= \text{length of the arc} + 2\ r$ and this means that $P= L + 2r$
Chapter 12  **SECTION 4: MEASUREMENT OF AREA**

**Introduction**

Dear students, we are going to study about how to find and calculate areas of various shapes in this section. We must always remember that the unit for measuring area is a square. Examples of area are \( cm^2, m^2, \text{ etc} \)

**Section objectives**

By the end of this section, you will be able to calculate for the;

- Area of a circle
- Area of a given sector
- Area of a given segment

---

Chapter 13  **Area of A Circle**

The area of a circle with a particular radius is given by

\[
A = \pi r^2
\]

Or

\[
A = \frac{1}{4} \pi d^2 \quad \text{where} \quad \pi = 3.142 \quad \text{and} \quad d = \text{diameter} \quad (d=2r)
\]

**EXAMPLE 1**

Find the area of a circle with radius 7cm. (Take \( \pi = \frac{22}{7} \))

**Solution**

\[
A = \pi r^2 \quad \text{where} \quad r=7 \quad \text{and} \quad \pi = \frac{22}{7}
\]

\[
A = \frac{22}{7} \times 7^2
\]

\[
A = 22 \times 7
\]

\[
= 154cm^2
\]
EXAMPLE 2

The area of a circle is \(154\text{ cm}^2\). Find its radius and circumference.

Solution

i. Radius

\[ A = \pi r^2 \] but \( A = 154, \quad \pi = \frac{22}{7} \]

Substituting

\[ 154 = \frac{22}{7} r^2 \]

\[ 154 \times 7 = 22r^2 \]

\[ r^2 = \frac{1078}{22} \]

\[ r = \sqrt{49} \]

\[ r = 7\text{ cm} \]

ii. The circumference of a circle \( (C) = 2\pi r \)

Where \( \pi = 3.142 \) or \( \frac{22}{7} \) and \( r = \text{radius} \)

\[ C = 2 \times 3.142 \times 7 \]

\[ = 14 \times 3.142 \]

\[ = 43.988\text{ cm} = 44\text{ cm} \]

Check your progress

1. Find the area of a semi-circle whose radius is 7cm. (Take \( \pi = \frac{22}{7} \))
2. Find the area of a circle whose circumference is 43.988
3. Find the area of a circle whose radius is 18cm.

Suggested Answers

1. 154 divided by 2 since it’s a semi-circle = \(77\text{ cm}^2\)
2. \( r = 7\text{ cm} \) therefore area of the circle = \(154\text{ cm}^2\)
3. \(1018\text{ cm}^2\)
Chapter 14 Area of A Sector

The Area of a sector of a circle with centre angle $\theta$ is given by

\[ A = \frac{\theta}{360} \times \text{Area of a circle} \]

\[ A = \frac{\theta}{360} \times \pi r^2 \]

EXAMPLE

Find the area of a sector that subtends an angle of 40° at the centre of circle with radius 8cm. (Take $\pi = \frac{22}{7}$)

Solution

\[ A = \frac{\theta}{360} \times \pi r^2 \]

Where $\theta = 40^\circ$, $\pi = \frac{22}{7}$ and $r = 8$

\[ = \frac{40}{360} \times \frac{22}{7} \times 8^2 \]

\[ = \frac{1}{9} \times 201.143 \]

\[ = 22.35 cm^2 \]

Example 2

Find the area of the sector that has radius 7.5 with an angle of 150°.

Solution

The area of a sector is given by

\[ A = \frac{\theta}{360} \times \pi r^2 \]

Where $\theta = 150^\circ$, $\pi = \frac{22}{7}$ and $r = 7.5$

\[ = \frac{150}{360} \times \frac{22}{7} \times 7.5^2 \]

\[ = \frac{5}{12} \times 176.786 \]

\[ = 73.66 cm^2 \]
Check your progress

1. Find the area of sector OAB given that angle AOB is 60 and the radius is 7cm.
2. The area of the circle is 48 cm\(^2\). Find the area of its sector.

Chapter 15 Area of A Segment

The segment is the area formed by a chord and an arc. Consider the diagram below;

The shaded portion is the minor segment while the other portion is the major segment

The area of the minor segment is given by

Area of a sector POQ - Area of the triangle OPQ

\[
= \frac{\theta}{360} \times \pi r^2 - \frac{1}{2} r^2 \sin \theta
\]

Example 1

In the diagram below, MN is a chord of a circle of radius 5cm. the chord subtends an angle of 96\(^\circ\) at the centre. Find the area of the minor segment cut off by MN.
Solution

Area of a segment = \( \frac{\theta}{360} \times \pi r^2 - \frac{1}{2} r^2 \sin \theta \)

\[
\begin{align*}
&= \frac{96}{360} \times \frac{22}{7} \times 5^2 - \frac{1}{2} \times 5^2 \sin 96 \\
&= 20.95 - 12.43 \\
&= 8.52 \text{ cm}^2
\end{align*}
\]

Example 2

Calculate the area of the shaded segment of the sector in the diagram below. (Take \( \pi = \frac{22}{7} \))
Solution

Area of a segment = \( \frac{\theta}{360} \times \pi r^2 - \frac{1}{2} r^2 \sin \theta \)

= \( \frac{75}{360} \times \frac{22}{7} \times 12^2 - \frac{1}{2} 12^2 \sin 75 \)

= 94.29 – 69.55

=24.74cm\(^2\)

Check your progress

1. A chord PQ of a circle of radius 5cm subtends an angle of 70\(^\circ\) at the center, O. find correct to 3 significant figures
   a) The length of the chord PQ
   b) The length of the arc PQ
   c) The area of the sector POQ
   d) The area of the minor segment cut off by PQ

2. Six horse are tethered with ropes measuring 14m each to the four corners of a rectangular grass land 21m \(\times\) 24m in dimension. Find;
   a) The maximum area that can be grazed by the horses
   b) The area of the grass that remains ungraced.

3. A quadrant of a circle has radius 10cm. Calculate
   a) Perimeter of the quadrant
   b) Area of the quadrant

Suggested Answers

3. i. Length of its arc=15.71cm
   hence perimeter of quadrant = 1 + 2r = 15.71 + 20= 35.71 cm

ii. Area of the quadrant= 78.55 cm\(^2\)

Summary

In this section, we learnt that;

- The area of a circle with a particular radius is given by \( A = \pi r^2 \) Or \( A = \frac{1}{4} \pi d^2 \) where \( \pi = 3.142 \) and \( d = \text{diameter} \) (d=2r)
• The Area of a sector of a circle with centre angle $\theta$ is given by $A = \frac{\theta}{360} \times \text{Area of a circle}$ and this is the same as $A = \frac{\theta}{360} \times \pi r^2$

• The segment is the area formed by a chord and an arc.

• The area of a segment is calculated as Area of a sector - Area of the triangle $= \frac{\theta}{360} \times \pi r^2 - \frac{1}{2} r^2 \sin \theta$
Chapter 16 SECTION 5: MEASUREMENT OF VOLUME

Introduction

Dear students, we are going to study about how to find and calculate the volume, surface areas and total surface area of various shapes in this section. We must always remember that the unit for measuring volume is a cube. Examples of the unit for measuring volume are $cm^3$, $m^3$, etc.

Section objectives

By the end of this section, you will be able to calculate for the;

- Volume, curved surface area and total surface area of a cylinder
- Volume, curved surface area and total surface area of a cone
- Area, volume, surface area and total surface area of a cube
- Volume, surface area and total surface area of a cuboid
- Volume, curved surface area and total surface area of pyramids

Chapter 17 Cylinder

A cylinder is a plane figure that has a curved surface and two parallel and congruent bases. This means that a cylinder has three surfaces; a top, bottom and middle. Most cylinders have circular base which means the top and bottom are circles. A cylinder can also be viewed as a prism with a circular cross-section. An Example of a cylindrical objects are milk or milo tin.
• **Volume**

To find the volume is to find the amount of space within the cylinder. First, find the area of the two circular shapes at the top and bottom and multiply it by the height of the cylinder. The two circular bases are circular hence their area is $\pi r^2 = \pi r^2$

Multiply by the height $= \pi r^2 \times h$

$= \pi r^2 h$

• **Curved Surface Area**

To find the curved surface area is to find the area of the middle part of the cylinder. This does not include the two circular shapes. First cut the cylindrical can up its wall; from the bottom to the top and spread it open. When it’s completely opened, it forms a rectangular shape hence the area of the rectangle becomes easier to find.

The top of the original cylinder is a circle hence the distance around the circle is called its circumference but we know circumference $(C) = 2\pi r$

When the cylinder wall is completely opened, this circumference of the circular end now becomes the length of the final rectangle. The dimensions of the rectangle are the circumference $(2\pi r)$ and the height $(h)$

So, the area $= \text{length} \times \text{width}$

$= \text{circumference} \times \text{height}$

$= 2\pi r \times h$

$= 2\pi rh$

• **Total Surface Area**

To find the total surface area of the cylinder, add the three surfaces together; the top, bottom and middle (the lateral surface area)

$= \text{top} + \text{bottom} + \text{middle}$

$= \pi r^2 + \pi r^2 + 2\pi rh$

$= 2\pi r^2 + 2\pi rh$

Factorizing $= 2\pi r (r + h)$
Example 1

A water tank with one end open has diameter 140mm and height 50mm. calculate

i. Area of the base
ii. Curved surface area
iii. Total surface area
iv. Volume of the tank.

Solution

Given the diameter of 140mm, radius= - \frac{1}{2} of the diameter= - \frac{1}{2} \times 140 = 70mm

i. Area of the base = \pi r^2
   = \frac{22}{7} \times 70^2
   = \frac{22}{7} \times 4900
   = 15400mm^2

ii. Curved surface area = 2\pi rh
   = 2 \times \frac{22}{7} \times 70 \times 50
   = 2 \times \frac{22}{7} \times 3500
   = \frac{22}{7} \times 7000
   = 22000mm^2

iii. Total surface area = \pi r^2 + 2\pi rh
    = 15400mm + 22000mm
    = 2 \times 3.142 \times 7 \times (7+15)
    = 37400mm^2

iv. Volume of the cylinder = \pi r^2 h
    = \frac{22}{7} \times 70^2 \times 50
    = \frac{22}{7} \times 4900 \times 50
    = \frac{22}{7} \times
    = \frac{22}{7} \times 245000
    = 770,000mm^3
Example 2

A closed cylinder has radius 7cm and height 15cm. find

a) The area of the curved surface
b) The area of its base
c) The total surface area
d) The volume of the cylinder

Solution

Diagram

\[ v. \text{ Curved surface area} = 2\pi rh \]
\[ = 2 \times 3.142 \times 7 \times 15 \]
\[ = 210 \times 3.142 \]
\[ = 659.82 \text{cm}^2 \]

\[ vi. \text{ Area of the base} = \pi r^2 \]
\[ = 3.142 \times 7^2 \]
\[ = 3.142 \times 49 \]
\[ = 153.958 \text{cm}^2 \]

\[ vii. \text{ Total surface area} = 2\pi r^2 + 2\pi rh \]
\[ = 2\pi r (r + h) \]
\[ = 2 \times 3.142 \times 7 (7+15) \]
\[ = 2 \times 3.142 \times 154 \]
\[ = 967.736 \text{cm}^2 \]

\[ viii. \text{ Volume of the cylinder} = \pi r^2 h \]
\[ = 3.142 \times 49 \times 15 \]
\[ = 2309.37 \text{cm}^2 \]

Check Your Progress

1. A closed cylinder has radius of 3 cm and height 7 cm. Find its total surface area. (Take \( \pi = \frac{22}{7} \))

2. A market woman uses a milo tin with one end open to measure the amount of gari to be sold. If the milo tin has a diameter of 140mm and height 50mm. calculate the

i. Area of the base of the tin
Chapter 18 Cone

A cone is a type of geometric shape. There are different kinds of cone. They all have flat surfaces on one side that tapers to a point on the other side. In order to calculate the surface area and volume of the cone, we first need to know these;

(A) RADIUS (r) is the distance from the center of the circle to the edge of the circle at the end
(B) HEIGHT (h) is the distance from the center of the circle to the tip of the cone
(C) SLANT HEIGHT (L) is the length from the edge of the circle to the tip of the cone.
(D) PI (π) is a special number approximately equal to 3.142 or $\frac{22}{7}$

**Volume**

Volume is how much space takes up the inside of cone. The answer is always in cubic unit.

Since the base of the cone is a circle, it has a radius(r). When a line is drawn from the apex of the cone, it forms a height (h). The circular base of the circle has an area $= \pi r^2$

Multiply by the height of the cone $\pi r^2 \times h$

Hence the volume of the cone is supposed to be $\pi r^2 h$

But a cone has a volume which is one-third of a cylinder.
The volume of a cone $= \frac{1}{3}$ of its base area $\times$ height

$= \frac{1}{3} \pi r^2 h$

- **Curved Surface Area**
  To find the curved surface of a cone, multiply the base radius of the cone by pi ($\pi$)
  $\pi \times r = \pi r$
  Now, multiply your answer by the length of the side of the cone (slant height)
  $= \pi r \times l$
  $= \pi rl$

- **Total Surface Area**
  To find the total surface area of a cone, note that there are two forms of cones that you can face with;
  a) **Solid cone;** cone with its base covered
  b) **Hollow cone;** cone without a base
  To find the total surface area of the cone, add the area of the base to the curved surface area but the area of the base the area of the circle $= \pi r^2$
  Hence;
  The total surface area of a hollow cone $= \pi rl$
  The total surface area of a solid cone $= \pi r^2 + \pi rl = \pi r (r + l)$

**Example 1**

The slant height and the height of a cone is 40 cm. If the radius of the cone is 32 cm. Calculate

a) The volume of the cone
b) The curved surface area

c) The total surface area

Solution

a) Volume of a cone \[ V = \frac{1}{3} \pi r^2 h \]
\[ = \frac{1}{3} \times 3.142 \times 32^2 \times 40 \]
\[ = 42898.77 \text{ cm}^3 \]

b) The curved surface area \( = \pi rl \)
\[ = 3.142 \times 32 \times 40 \]
\[ = 3.142 \times 1280 \]
\[ = 4021.76 \text{ cm}^2 \]

c) The total surface area \( = \pi r^2 + \pi rl \)
\[ = \pi r (r + l) \]
\[ = 3.142 \times 32 (32 + 40) \]
\[ = 3.142 \times 32 (72) \]
\[ = 7239.168 \text{ cm}^2 \]

OR

Total surface area \[ = \pi r^2 + \pi rl \]
\[ = 3.142 \times 32^2 + (4021.76) \text{ in b} \]
\[ = 3217.408 + 4021.76 \]
\[ = 7239.168 \text{ cm}^2 \]

Example 2

A conical tent is of the diameter 24 m at the base and its height is 16 m.

a) Find the slant height

b) The canvas required in square meters

c) At most how many persons can the tent accommodate if each person required 54 m\(^3\) of air? (Take \( \pi = \frac{22}{7} \))

Solution

Ask 1: What data is given in the problem?

The diameter of the cone = 24 m
The height of the cone = 16 m

The volume of air required by each person = 54 m³

**Ask 2:** List the things to be determined.

The slant height of the cone

The area of the canvas

Number of persons accommodated

**Ask 3:** List the steps of calculation. To determine the slant height, use the formula. To determine the area of canvas, determine the curved surface area of the cone. To determine the number of persons, calculate the volume of cone and divide by the volume of air required by each person.

Hence,

A) Slant height, \( l = \sqrt{r^2 + h^2} \)

\[ l = \sqrt{12^2 + 16^2} \]

\[ = \sqrt{144 + 256} \]

\[ = \sqrt{400} \]

\[ = 20 \text{ m} \]

B) The canvas required = Curved surface area of cone = \( \pi rl \)

\[ = \frac{22}{7} \times 12 \times 20 \]

\[ = 754.3 \text{ sq. m} \]

d) Volume of air in the tent = Volume of a cone = \( \frac{1}{3} \pi r^2 h \)

\[ = \frac{1}{3} \times \frac{22}{7} \times 12^2 \times 16 \]

\[ = \frac{16896}{7} \text{ m}^3 \]

Volume of air required by one person = 54 m³

No. of persons that can be accommodated

\[ = \frac{\frac{16896}{7}}{54} \]

\[ = 44.7 \]
=45 persons

Check Your Progress

1. A cone is 14cm deep and the base radius is 4.5cm. calculate the volume of water that is exactly half the volume of the cone. (Take $\pi = \frac{22}{7}$)

2. The height of a right circular cone is 4cm. the radius of its base is 3cm. find its curved surface area. (Take $\pi = \frac{22}{7}$)

3. A sector containing an angle of 120º is cut from a circle of radius 21cm and folded into a cone. Find
   a) Curved surface area of the cone
   b) Total surface area of the cone

4. Let O and C be the center of the base and the vertex of a right circular cone. Let B be any point on the circumference of the base. If the radius of the cone is 6cm and if angle OBC =60, find
   a) The height of the cone
   b) The curved surface area of the cone

5. Find the total surface area of a cone of base radius 4cm and with slant height 9cm.

Suggested Answers

1. 148.5 cm³
2. 47.14 cm²
3. 50.272 + 113.112
   = 163.384 cm²

Cube

A cube is a three-dimensional figure with six faces. All its angles are right –angled. All the faces of a cube are equal. It is a shape covered by six squares. In a cube, all the lengths are equal since it is formed from a square.
Area

To find the area of the face of a cube, multiply the length by the other length = \( length \times length \)

= \( l \times l \)

= \( L^2 \)

Volume

The cube is a special case of a cuboid, where \( l = b = h \). Hence, volume of cube = \( L \times L \times L = L^3 \)

Total Surface Area

Add up the area of all the six square faces.

= \( L^2 + L^2 + L^2 + L^2 + L^2 + L^2 \)

= \( 6L^2 \)

Example 1

If the volume of a cube is 2197, find;

a) length of one of its sides
b) surface area.

Solution

i. Let the edge (length) of the cube be \( x \) cm.
So, its volume = $a \ cm^3$

Therefore, from the question, we have

$a \ cm^3 = 2197$

$a \ cm^3 = 13 \times 13 \times 13$ or $\sqrt[3]{2197}$

So, $a = 13$

Therefore, the length of the cube = 13 cm

ii. Now, surface area of the cube = $6a^2$

$= 6 \times 13 \ cm \times 13 \ cm$

$= 1014 \ cm^2$

Thus, surface area of the cube is $1014 \ cm^2$

**Check your progress**

1. Find the volume of the following cubes
   a) with a side 4 cm
   b) with a side 1.5 m
2. Find the volume of a cube whose length is 5 cm.
3. The volume of a cube is $8 \ cm^3$. Find the length of one of its sides.
4. The volume of a cube is $64 \ cm^3$. Find its total surface area

**Suggested Answers**

1. a) $4 \times 4 \times 4 = 64 \ cm^3$
   
   b) $1.5 \times 1.5 \times 1.5 = 3.375 \ cm^3$

2. $125 \ cm^3$

3. 2 cm

4. 96 cm$^3$
Chapter 19 Cuboid

A cuboid is also a three-dimensional object with six faces. All its angles are right-angled and opposite faces are equal. A cuboid is the same as a rectangle prism or a rectangular solid. In a cuboid, the length, width and height are different.

**Volume**

To find the volume of a cuboid, multiply the dimensions; length, width and height.

\[
\text{Volume} = \text{length} \times \text{width} \times \text{height}
\]

\[V = lwh\]

**Surface Area**

To calculate the surface area of the cuboid, we first calculate the area of each of the six faces and add up all the area to get the total surface area.

First, find the area of the rectangle at the top and bottom surface. The top and bottom have only length and breadth, hence its area is calculated by multiplying the length by the breadth = \(l \times b = lb\)

\[\text{TOP} \quad \text{BOTTOM}\]

\[lb + lb = 2lb\]

Secondly, find the area of the front and back surface. The front and back has only length and height = \(l \times h = lh\)

\[\text{FRONT} \quad \text{BACK}\]

\[lh + lh = 2lh\]

Thirdly, find the area of the two other side surfaces. The two-side surface has only breadth and height = \(b \times h = bh\)
\[ bh + bh = 2bh \]

- **Total Surface Area**

Hence the total surface area is the addition of all the area of the six faces

\[ = 2lb + 2lh + 2bh \]

Factorizing \[ = 2 \left( lb + lh + bh \right) \]

Example 1

Find the height of a cuboid whose volume is 275 cm\(^3\) and base area is 25 cm\(^2\).

**Solution**

Volume of a cuboid = Base area \times Height

Hence height of the cuboid

\[ \frac{\text{Volume of cuboid}}{\text{Base area}} \]

\[ \frac{275}{25} \]

= 11 cm

Height of the cuboid is 11 cm

Example 2

Example 9: A key soap box is in the form of a cuboid of measures 60 m \(\times\) 40 m \(\times\) 30 m. How many cuboidal boxes can be stored in it if the volume of one box is 0.8 m\(^3\)?

**Solution**

Volume of one box = 0.8 m\(^3\)

Volume of key soap box = 60 \(\times\) 40 \(\times\) 30 = 72000 m\(^3\)

Number of boxes that can be stored in the key soap box =

\[
\frac{\text{Volume of key soap box}}{\text{volume of one box}}
\]
Hence the number of cuboidal boxes that can be stored in the key soap box is 90,000.

Example 3
Consider a rectangular box with breadth 5cm and length 4cm. If the box has a height of 3cm, find

a) its total surface area
b) its volume

Solution

a) Total surface area of cuboid = \(2lb + 2lh + 2bh\)
   
   \[
   = 2(4(5) + 4(3) + 5(3))
   \]
   
   \[
   = 2(20+12+15)
   \]
   
   \[
   = 2(47)
   \]
   
   \[
   = 94 \text{ cm}^2
   \]

b) Volume = \(\text{length} \times \text{width} \times \text{height}\)

\[
V = lwh
\]

\[
= (4 \times 5 \times 3)
\]

\[
= 60 \text{ cm}^3
\]

Example 4
Find the volume of a cuboidal stone slab of length 3m, breadth 2m and thickness 25cm.

Solution

Here, \(l = 3\)m, \(b = 2\)m and \(h = \frac{25}{100} \text{ cm} = 0.25\text{m}\)

Note that here we have thickness as the third dimension in place of height) So, required
volume = l×b×h
= 3×2×1/4 m
= 1.5 cm$^3$

Check Your Progress

1. The base area of a rectangular box is 30 cm$^2$ and its height is 3 cm$^2$. Find its volume
2. The volume of a key soap box which is 4 cm long and 6 cm high is 72 cm$^3$. Find its breadth
3. Find the volume of a rectangular block 15 cm long, 5 cm wide and 10 cm long.
4. A cuboid of base 12.5 cm by 20 cm holds exactly 1 litre of water. What is the height of the cuboid? Note that 1000 cm$^3$ = 1 litre
5. The area of a rectangular floor is 13.5 m$^2$. One side is 1.5 m longer than the other. Calculate the dimensions of the floor.

Suggested answers

1. 90 cm$^3$
2. 3 cm
3. 750 cm$^3$
4. 4 cm
5. 3 m × 4.5 m

Sphere

A sphere is a perfectly round geometrical object that is three dimensional, with every point on its surface equidistant from its center. Many commonly-used object such as balls, globes are spheres.
It takes two cones of water to fill a sphere. But we know the volume of a cylinder is the area of the base multiplied by the height.

\[
\text{volume of a cylinder} = \pi r^2 h
\]

But it takes three times the volume of the cone to get the volume of a cylinder. This is given as;

\[
\text{volume of a cone} = \text{volume of cylinder}/3
\]

\[
= \frac{\pi r^2 h}{3}
\]

\[
= \frac{1}{3} \pi r^2 h
\]

So, since it takes two cones to fill the sphere, the height of the cone is equal to the height of the corresponding cone which is also equal to the height of the sphere. So, adding the volume of the two cones will be equal to the volume of the sphere. That is

\[
\text{one cone} + \text{one cone} = \text{sphere}
\]

\[
\frac{1}{3} \pi r^2 h + \frac{1}{3} \pi r^2 h = \frac{\pi r^2 h}{3} + \frac{\pi r^2 h}{3}
\]

\[
= \frac{2 \pi r^2 h}{3}
\]

The \( r + r \) = height and the height in a sphere is also equal to the diameter

Which is \( r + r = \text{diameter} \)

\[ 2r = \text{diameter} \]

Since the diameter (2r) is equal to the height, substitute 2r in place of h

\[
= \text{sphere} = \frac{2 \pi r^2 h}{3} \quad \text{but} \quad h=2r
\]

\[
= \frac{2 \pi r^2 (2r)}{3}
\]

\[
= \frac{4 \pi r^3}{3}
\]

Therefore, the volume of a sphere \( = \frac{4 \pi r^3}{3} \)

- **Curved surface area**

Archimedes discovered that a cylinder that circumscribes a sphere has a curved surface equal to
the surface area of the sphere. That is

Curved surface area of a sphere \[=\] curved surface of a cylinder
\[=\] \[2\pi rh\]

But the cylinder has height equal to that of the sphere. Therefore, the height of the sphere is equal to the diameter of the sphere. So, the height is equal to the sum of two radius which is mathematically represented by height

\[h = r + r\]
\[h = 2r\]

Now substituting \(h = 2r\) into the formula above is;

Curved surface area of a sphere \[=\] \[2\pi r(h)\]
\[=\] \[2\pi r(2r)\]
\[=\] \[2\pi r \times 2r\]
\[=\] \[4\pi r^2\]

Example 1

Find the curved surface area of a solid hemisphere that has a total surface of \(675\pi \text{ cm}^2\).

Solution

The total surface area of a hemisphere \[=\] \[3\pi r^2\]

From the question, the total surface area of that hemisphere is \(675\pi \text{ cm}^2\)
\[=\] \[3\pi r^2= 675\pi \text{ cm}^2\].
\[=\] \[r^2 = 225\text{cm}^2\].

The curved surface area of a solid hemisphere \[=\] \[2\pi r^2\]
\[=\] \[2 \times 3.142 \times 225\]
\[=\] \[450 \times 3.142\]
\[=\] \[1413\text{cm}^2\]

Check Your Progress
1. Find the area available if a hollow sphere in which a circus motorist performs his stunt has an inner diameter of 14m.
2. A solid circular cylinder has radius 14cm and height 8cm. finds its;
   a) Curved surface area
   b) Total surface area
3. The total surface area of a solid circular cylinder is 1540cm$^2$. If the height if five times the radius of the base, then find the height of the cylinder.
4. Calculate the surface area of a sphere of radius 7 cm
5. A hollow sphere was completely filled with 1 litre of oil. Calculate the internal radius of the sphere in cm

**Suggested Answers**

4. $616 \, cm^2$

5. 6.2 cm (2 significant figures)

---

**Chapter 20 Pyramid**

A pyramids are solids whose base is a polygon and the faces are triangles. There are three important parts in any pyramid namely; base, face and apex. The base of a pyramid may be of any shape. The faces of the pyramid are mostly isosceles triangles. All the triangular faces meet at a single point called the apex.

There are different types of pyramid which are named using their base shape of the pyramid. Some of these pyramids are below;

<table>
<thead>
<tr>
<th>TYPE OF SHAPE</th>
<th>NUMBER OF LATERAL FACES</th>
<th>B A S E Shape</th>
</tr>
</thead>
<tbody>
<tr>
<td>Triangular pyramid</td>
<td>3</td>
<td>triangle</td>
</tr>
<tr>
<td>Square pyramid</td>
<td>4</td>
<td>square</td>
</tr>
<tr>
<td></td>
<td>5</td>
<td>pentagon</td>
</tr>
</tbody>
</table>
To find the total surface area of any pyramid, you have to find the sum of the lateral area of its triangular faces and the area of the pyramid base.

To find the lateral area of the triangular faces of the pyramid, first of all find the area of one triangular face.

Area of one triangular face of the pyramid is \( \frac{1}{2} \times \text{base} \times \text{height} \)

Therefore, Area of one triangular face \( = \frac{1}{2} b h \)

Then after finding the area of a triangle, you multiply the result by the number of lateral triangular faces that is represented by a variable \( n \). So, area of lateral triangular face of the pyramid is equal to the area of one triangular face multiplied by the number of lateral faces represented by \( n \). Representing the formula mathematically is,

**Area of lateral triangular faces of \( n \) pyramid**  
\[ \text{area of one triangular face} \times \{n\} \text{ lateral faces} \]

Where \( \{n\} \) is the number of lateral faces \[ = \frac{1}{2} bh \times \{n\} \]
\[ = \frac{1}{2} bhn \]
Therefore, the total surface area of a pyramid is equal to the sum of area of lateral triangular faces and the area of the pyramid base.

Total surface area of a pyramid = area of the lateral triangular faces + area of pyramid base

**Volume of A pyramid**

Pyramids can be compared with prisms with the same base and height. But it takes three times the number of a pyramid to get the volume of a prism. So, if 3 pyramid = 1 prism

Then 1 pyramid = \( \frac{1}{3} \) of a prism

So, the volume of a pyramid is equal to the product of \( \frac{1}{3} \) of the area of base of the pyramid and the height.

Therefore,

Volume of a pyramid = \( \frac{1}{3} \) x area of the base x height

**Example 1**

The height of a pyramid on a square base is 15cm. If the length of a side of the base is 5cm, find the volume of the pyramid.

**Solution**

Volume of a pyramid = \( \frac{1}{3} \) x area of the base x height

Area of the base = 5cm x 5cm = 25\( cm^2 \)

Volume of a pyramid = \( \frac{1}{3} \) x 25 x 15

Volume of a pyramid = 125\( cm^3 \)

**Check Your Progress**

1. A pyramid has a square base with sides 8cm long. Its other faces are isosceles triangles with a vertical height of 10cm. Find
   a) Its volume
   b) The total surface area
2. The height of a right pyramid on an equilateral triangular base is 7.8 cm. If the length of a side of the base is 18 cm, find
   a) Volume
   b) Surface area

3. A right pyramid is on a square base of side 4 cm. The slanting edge of the pyramid is 3.5 cm. Calculate the volume of the pyramid.

4. The height of a pyramid on a square base is 15 cm. If the volume is 80 cm$^3$, find the length of the sides of the base.

**Suggested answers**

3. 56 cm$^3$

4. 4 cm

**Summary**

In the section 5 of this unit, we learnt that:

- A cuboid is a three-dimensional object with six faces and its opposite faces being equal.
- The volume of a cuboid is $\text{length} \times \text{width} \times \text{height} = \text{LWH}$ and its total surface area is $2(lb + lh + bh)$
- A cylinder is a solid figure that has a curved surface and two parallel and congruent bases.
- The volume of a cylinder is $2\pi r^2 h$, while the Curved Surface Area is $2\pi rh$ and the Total Surface Area is $2\pi r (r + h)$
- The volume of a cone is given as $\frac{1}{3} \pi r^2 h$, its curved surface area $= \pi rl$ while the total surface area is also given as $\pi r^2 + \pi rl$
- To find the area of the face of a cube, multiply the length by the other length
- The volume of a cube is also computed as $L \times L \times L$
- To calculate for the total Surface Area of the cube, add up the area of all the six square faces $= L^2 + L^2 + L^2 + L^2 + L^2 + L^2 = 6L^2$
- A sphere is a perfectly round geometrical object that is three dimensional, with every point on its surface equidistant from its center.
- Volume of a pyramid $= \frac{1}{3} \times \text{area of the base} \times \text{height}$
- The volume of a sphere is $\frac{4}{3} \pi r^3$, its curved surface area is $= 4\pi r^2$
- Pyramids are solids whose bases are polygons and the faces are triangles
- Area of lateral triangular faces of n pyramid $= \text{area of one triangular face} \times \{n\}$
lateral faces = \( \frac{1}{2}bh \times \{n\} = \frac{1}{2}bhn \)
SECTION 6: MATHEMATICAL VOCABULARY ASSOCIATED WITH MENSURATION

Introduction

Good job! You have finally made it to the last section of unit 4. We want to conclude the entire unit in this section by looking at some of the mathematical vocabulary that are key to the unit on mensuration. Do well to go through all of these terms with us.

Section objectives

By completing this unit, you will be able to:

• Explain briefly each of other major terminologies in mensuration
• Sketch diagrams to represent the major terminologies in mensuration

Mensuration: It is a branch of mathematics which deals with the lengths of lines, areas of surfaces and volumes of solids.

Plane Mensuration: It deals with the sides, perimeters and areas of plane figures of different shapes.

Solid Mensuration: It deals with the areas and volumes of solid objects.

Rectangle: A rectangle is a plane; whose opposite sides are equal and diagonals are equal. Each angle is equal to 90°

Square: A square is a plane figure bounded by four equal sides having all its angle as right angles

Solid: Bodies which have three dimensions in space are called solid. For example, a block of wood. A body, which has the three dimensions length, breadth and height, is a solid, whereas a body with only two dimensions (length and breadth) is not a solid.

Prism: A prism is a solid, bounded by plane faces of which two opposite sides known as bases are parallel and congruent polygons.

Base: The congruent and parallel faces of a prism are called its bases. The other faces of a prism can be either oblique to the faces or perpendicular to them.

Right prism: A right prism is a prism in which lateral sides are rectangular or perpendicular to their bases.

Lateral faces: The side faces of a prism are called its lateral faces.

Lateral surface area: The sum area of all the lateral faces of a prism is called its lateral surface area.
**Pyramid:** A solid of triangular lateral sides having a common vertex and plane rectilinear bases with equal sides is called pyramid.

**Height of the pyramid:** The length of the perpendicular drawn from the vertex of a pyramid to its base is called the height of the pyramid. The side faces of pyramid form its lateral surface.

**Regular pyramid:** If the base of a pyramid is a regular figure i.e., a polygon with all sides equal and all angles equal, then it is called a regular pyramid.

**Right pyramid:** If the foot of the perpendicular from the vertex of a pyramid to its base is the centre of the base then it is called a right pyramid.

**Slant height of a regular right pyramid:** The slant height of a regular right pyramid is the length of the line segment joining the vertex to the mid-point of one of the sides of the base.

**Tetrahedron:** When the base of a right pyramid is a triangle, then it is called a tetrahedron.

**Regular tetrahedron:** A right pyramid with equilateral triangle as its base is called a regular tetrahedron.

**Check your progress**

To help you remember the meaning of these terminologies, go through this activity with us;

**Activity 1**

i. Pick a sheet of paper
ii. Write each of the mathematical terminologies associated with mensuration vertically and well-itemized on the sheet
iii. Put your course manual aside
iv. Try to define or describe each of the terminologies noted on your sheet.
v. Compare your responses to the lesson in section 6 of this unit 4.

**Activity 2**

Make a sketch to represent each of the terminologies learnt.

Congratulations for going through the activity with us.

**Suggested answer**

Refer to the lesson on unit 4 section 6
Summary

In brief, we have studied in this section of unit 4 of the course, that the concept of mensuration is imbedded with lot of terminologies which we need to have at finger tips as student teachers. Knowing these terminologies helps us to be able to pose and also apply them in solving real life problems that involve the application of mensuration in teaching real life phenomena. We hope after painstakingly going through the entire unit with us, you have now come to agree with us that indeed the concept of mensuration is not meant for elective mathematics students only because of its applicability in our daily life activities. You are encouraged once again to go through the remaining units for more interesting concepts of this course.

REFERENCES


UNIT 5 GLOBAL MATHEMATICS AND INTRODUCTORY STATISTICS (PATTERNS IN DATA)

1. The earth as a sphere and measuring latitudes
2. Measuring longitudes and solving problems
3. Teaching ways of collection of data
4. Teaching measures of central tendencies and locations
5. Teaching measures of dispersion
6. Teaching graphical representation (cumulative frequency)

UNIT INTRODUCTION

Dear Learners,

In this unit, we are going to learn the basic concepts in global mathematics and introductory statistics. Therefore, our learning indicators are:

1. the earth as a sphere and measuring latitudes
2. measuring longitudes and solving problems
3. teaching ways of collection of data
4. teaching measures of central tendencies and locations
5. teaching measures of dispersion
6. teaching graphical representation (cumulative frequency).

Take your resources such as ruler, pencil, calculator and paper as we explore these concepts.
UNIT 5 SECTION 1 THE EARTH AS A SPHERE

INTRODUCTION

In bearing, we learned that it is a clockwise angular movement between two distance places. And the three rules to using bearings taking reading in bearing starts from the North Pole in clockwise direction and ends also at the North pole (i.e., North – North Pole reading), all angles formed while taking reading in bearing is equal to 360 degrees, and all questions in bearing leads into the formation of a triangle. In this section, our learning indicators are:

- Explain the concept of the earth as a sphere and the great circle.
- Explain the concept of the latitude and measure distances.

The Concept of the Earth as a Sphere

The Earth is very close to a sphere (ball) shape, with an average radius of 6,371 km. It is actually a bit flat at the poles, but only by a small amount. This is part of Earth Geometry, which is a special case of spherical geometry.
When we measure distances that a boat or aircraft travels between any two places on the Earth, we do not use straight line distances, since we need to go around the curve of the Earth from one place to another. This is going to be a lot less than the distance a plane flies around the surface of the Earth.

A great circle is defined as any circle drawn on a globe (or another sphere) with a centre that includes the centre of the globe. Thus, a great circle divides the globe into two equal halves. Since they must follow the circumference of the Earth to divide it, great circles are about 40,000 km (24,854 miles) in length along meridians. At the equator, great circle is a little bit longer as the Earth is not a perfect sphere.

In addition, great circles represent the shortest distance between two points anywhere on the Earth's surface. Because of this, great circles have been important in navigation for hundreds of years but their presence was discovered by ancient mathematicians.

Also, great circles are easily based on the lines of latitude and longitude. Each line of longitude, or meridian, is the same length and represents half of a great circle. This is because each meridian has a corresponding line on the opposite side of the Earth. When combined, they cut the globe into equal halves, representing a great circle. For example, the Prime Meridian at 0° is half of a great circle. On the opposite side of the globe is the International Date Line at 180°. It too represents half of a great circle. When the two are combined, they create a full great circle which cuts the Earth into equal halves.

Furthermore, the only line of latitude, or parallel, characterized as a great circle is the equator because it passes through the exact centre of the Earth and divides it in half. Lines of latitude north and south of the equator are not great circles because their length decreases as they move toward the poles and they do not pass through Earth's centre. As such, these parallel lines are considered small circles.

The commonest use of great circles is for navigation because they represent the shortest distance between two points on a sphere. Due to the earth’s rotation, sailors and pilots using great circle routes must constantly adjust their route as the heading changes over long distances. The only places on Earth where the heading does not change is on the equator or when traveling due north or south. Because of these adjustments, great circle routes are broken up into shorter lines, which cross all meridians at the same angle.

**The Concept of the Latitude**

Latitude is a measurement on a globe or map of location north or south of the Equator. Technically, there are different kinds of latitude—geocentric, astronomical, and geographic (or geodetic)—but
there are only minor differences between them. In most common references, geocentric latitude is implied.

Given in degrees, minutes, and seconds, geocentric latitude is the arc subtended by an angle at Earth’s centre and measured in a north-south plane from the Equator. Thus, a point at $30^\circ 15'20'' N$ subtends an angle of $30^\circ 15'20''$ at the centre of the globe.

The arc between the Equator and either geographic pole is $90^\circ$ (one-fourth the circumference of the Earth, or $\frac{1}{4} \times 360^\circ$), and thus the greatest possible latitudes are $90^\circ N$ and $90^\circ S$. As aids to indicate different latitudinal positions on maps or globes, equidistant circles are plotted and drawn parallel to the Equator and each other, known as parallels of latitude.

Geographic latitude, which is the kind used in mapping, is calculated using a slightly different process. Because Earth is not a perfect sphere (i.e. the planet’s curvature is flatter at the poles), geographic latitude is the arc subtended by the equatorial plane and the normal line that can be drawn at a given point on Earth’s surface. The normal line is perpendicular to a tangent line touching Earth’s curvature at that point on the surface.

Different methods are used to determine geographic latitude, namely by taking angle-sights on certain polar stars or by measuring with a sextant the angle of the noon Sun above the horizon. The length of a degree of arc of latitude is approximately $111 \text{ km (69 miles)}$, varying, because of the non-uniformity of Earth’s curvature, from $110.567 \text{ km (68.706 miles)}$ at the Equator to $111.699 \text{ km (69.41 miles)}$ at the poles.

Authalic Latitude is based on a spherical earth:

- Measures the position of a point on the earth's surface in terms of the angular distance between the equator and the poles.
- Indicates how far north or south of the equator a particular point is situated.
- **North latitude**: all points north of the equator in the northern hemisphere
- **South latitude**: all points south of the equator in the southern hemisphere

- Latitude is measured in angular degrees from $0^\circ$ at the equator to $90^\circ$ at either of the poles.
  - A point in the northern hemisphere $40$ degrees north of the equator is labeled Lat. $40^\circ N$.
  - Forty degrees south of the equator, the label changes to Lat. $40^\circ S$.

- The north or south measurement of latitude is actually measured along the meridian which passes through that location.
• It is known as an arc of the meridian.

• Geodetic Latitude is based on an ellipsoidal earth:
  o The ellipsoid is a more accurate representation of the earth than a sphere since it accounts for polar flattening.
  o Modern large-scale mapping, GIS, and GPS technology all require the higher accuracy of an ellipsoidal reference surface.

• When the earth’s shape is based on the WGS 84 Ellipsoid:
  o The length of 1° of latitude is not the same everywhere as it is on the sphere.
  o At the equator, 1° of latitude is 110.57 km (68.7 miles).
  o At the poles, 1° of latitude is 111.69 km (69.4 miles).

**Example 1**

1. The equator can also be called the:

   A. Prime Meridian
   B. Parallel of Latitude
   C. Great Circle
   D. Both 1 and 2
   E. Both 2 and 3

2. Which of the following is not true of parallels of latitude?

   A. They are true east-west lines
   B. Any two are always equal distances apart
   C. Always meet at the poles
   D. Related to the x-axis of the Cartesian coordinate system

3. Which of the following is not true of meridians of longitude?
A. They always meet at the poles

B. True north-south lines

C. Each is equal to half the length of a great circle

D. Always begin with the Prime Meridian through Greenwich, England

**Measuring Latitude and Distances**

Parallels of latitude decrease in length with increasing latitude. Mathematically, the length of parallel at latitude \( x \) is 
\[
\cos(x) \times (\text{length of equator})
\]

The length of each degree is obtained by dividing the length of that parallel by 360°. For instance, the cosine of 60° is 0.5. So the length of the parallel at that latitude is one half the length of the equator.

Since the variation in lengths of degrees of latitude varies by only 1.13 km (0.7 miles), the standard length of 111.325 km (69.172 miles) is used. For instance, anywhere on the earth, the length represented by 3° of latitude is 3 \times 111.325 km or 333.975 km.

When either latitude or longitude are expressed in degrees, minutes and seconds, the number of degrees is followed by the symbol °, the number of minutes is followed by the symbol ′, and the number of seconds is followed by the symbol ″. Thus, 32°44′10″ means 32 degrees, 44 minutes and 10 seconds. (Since there are 60 minutes in a degree and 60 seconds in a minute, neither the number of minutes nor the number of seconds can ever exceed 60.)

Therefore, it is easy to convert latitude or longitude expressed in degrees, minutes and seconds to decimal form since
\[
\text{minute} = \frac{1}{60} \text{ degree} \quad \text{and} \quad \text{second} = \frac{1}{60} \times \frac{1}{60} \text{ degree}
\]

The Radius (R) of the Earth is 6400 km

**Example 2**

1. Change 45°24′30″ to a decimal notation in degrees.

**Solution**
2. Change $22^\circ 54'17''$ to a decimal notation in degrees.

**Solution**

$$22^\circ 54'17'' = 22 + \left( 54 \times \frac{1}{60} \right) + \left( 17 \times \frac{1}{60} \times \frac{1}{60} \right) = 22.904722$$

3. Change $455.4083333$ to degree, minutes, seconds.

**Solution**

We know the first entry will be $45^\circ$.

Find the number of minutes in the remaining $0.4083333$ by dividing by $\frac{1}{60}$. i.e. $45.408333 \div \frac{1}{60} = 45.408333 \times 60 = 24$, with $0.4999998$ of a minute remaining.

Find the number of seconds in the remaining $0.4999998$ minutes by dividing by $\frac{1}{60}$. i.e. $0.4999998 \div 1/60 = 29.999988$, or 30 seconds.

Therefore, $455.4083333 = 45^\circ 24'30''$

This conversion missed by a slight amount on the exact seconds because $455.4083333$ was an approximation of the original $45^\circ 24'30''$.

**Activity 5.1**

1. Describe the relationship between the Cartesian coordinate system and the geographic grid.

2. Convert $35.40^\circ$ into degrees, minutes, and seconds.

3. Explain why the length of a degree of longitude decreases as one approaches the poles.
• The Earth is very close to a sphere (ball) shape, with an average radius of 6,371 km.
• A great circle is defined as any circle drawn on a globe (or another sphere) with a centre that includes the centre of the globe.
• Latitude is a measurement on a globe or map of location north or south of the Equator.
• Different methods are used to determine geographic latitude, namely by taking angle-sights on certain polar stars or by measuring with a sextant the angle of the noon Sun above the horizon.
• The length of a degree of arc of latitude is approximately 111 km (69 miles), varying, because of the non-uniformity of Earth’s curvature, from 110.567 km (68.706 miles) at the Equator to 111.699 km (69.41 miles) at the poles.
• Geographic latitude can also be measured in degrees, minutes, and seconds.
• The length of each degree is obtained by dividing the length of that parallel by 360°.
• It is easy to convert longitude or latitude expressed in degrees, minutes and seconds to decimal form since 1 minute = \( \frac{1}{60} \) degree and 1 second = \( \frac{1}{60} \times \frac{1}{60} \) degree.
INTRODUCTION

In the previous section, we learned that the Earth is very close to a sphere, with an average radius of $6,371\text{ km}$, the great circle is any circle drawn on a globe, latitude is a measurement on a globe or map of location north or south of the Equator, and $1\ \text{ minute} = \frac{1}{60}\ \text{ degrees}$ or $1\ \text{ second} = \frac{1}{60} \times \frac{1}{60}\ \text{ degrees}$.

In this section, our learning indicators:

- Explain the concept of the longitude
- Discuss how to measure longitudes and distance.

The Concept of the Longitude

Longitude is a measurement of location east or west of the prime meridian at Greenwich, the specially designated imaginary north-south line that passes through both geographic poles and Greenwich, London. In Ghana, towns such as Bawku and Tema are exactly located on the Greenwich.

Measured also in degrees, minutes, and seconds, longitude is the amount of arc created by drawing first a line from the Earth’s centre to the intersection of the Equator and the prime meridian and
then another line from the Earth’s centre to any point elsewhere on the Equator. Longitude is measured $180^\circ$ both east and west of the prime meridian.

As aids to locate longitudinal positions on a globe or map, meridians are plotted and drawn from pole to pole where they meet. The distance per degree of longitude at the Equator is about $111.32\ km (69.18\ miles)$ and at the poles, 0.

The longitude measures the position of a point on the earth’s surface east or west from a specific meridian, called the *prime meridian*. The following are some other characteristics:

- The longitude of a place is the arc, measured in degrees along a parallel of latitude from the prime meridian.
- The most widely accepted prime meridian is based on the *Bureau International de l'Heure (BIH) Zero Meridian*:
  - Defined by the longitudes of many BIH stations around the world.
  - The prime meridian has the angular designation of $0^\circ$ longitude.
  - All other points are measured with respect to their position east or west.
  - Longitude ranges from $0^\circ$ to $180^\circ$, either east or west.
- Since the placement of a prime meridian is arbitrary, countries often use their own.
  - For the purposes of measurement, no one prime meridian is better than another
  - Widely accepted meridian allows comparison between maps in different areas.
- The distance represented by a degree of longitude varies upon where it is measured.
  - The length of a degree of longitude along a meridian is not constant because of polar flattening.
  - At the equator, the approximate length is determined by dividing the earth’s circumference (24,900 miles) by 360 degrees: 111.05 kilometers (69 miles).
  - The meridians converge at the poles, and the distance represented by one degree decreases.
  - At $60^\circ$ N latitude, one degree of longitude is equal to about 55.52 kilometers (34.5 miles).
  - The Radius (R) of the Earth is $6400\ km$

**Measuring longitude and distance**
Because the earth is not a perfect sphere, the equatorial circumference does not equal that of the meridians.

On a perfect sphere, each meridian of longitude equals one-half the circumference of the sphere.

The length of each degree is equal to the circumference divided by 360.

Each degree is equal to every other degree.

Measurement along meridians of longitude accounts for the earth’s polar flattening:

Degree lengths along meridians are not constant:

- 111.325 km (69.172 miles) per degree at the equator
- 16.85 km (10.47 miles) per degree at 80°N
- 0 kilometres at the poles

The distance between meridians of longitude on a sphere is a function of latitude. Mathematically, Length of a degree of longitude is \( \cos(latitude) \times 111.325 \) km.

Since a difference in latitude or longitude of one second is quite small (less than 1/50 of a mile), then longitudes and latitudes of geographical locations like cities are usually expressed only in degrees and minutes, not seconds.

**Example 1**

1° of longitude at 40°N = \( \cos(40°) \times 111.325 \) or 85.28 km

These lengths are based on an ellipsoid and are similar to the lengths computed with the spherical formula.

When calculating distances over large areas, the authalic sphere can be used as a reference surface. The shortest distance between two points on a sphere is the arc on the surface directly above the true straight line.

The arc is based on a great circle.

The difference between the sphere and ellipsoid is important when working with large areas.
At a scale of 1:40,000,000, a 23 km error in distance would equal a pen line (0.5 mm) on paper.

**Example 2**

1. Two towns \( P(42^\circ W, 30^\circ N) \) and \( Q(18^\circ E, 30^\circ N) \) are on the surface of the earth. Find the time it will take a pilot flying at \( 600 km h^{-1} \) to travel from \( P \) to \( Q \).

**Solution**

The Earth Surface

\[
\theta = 42^\circ + 18^\circ = 60^\circ
\]

The Radius (R) of the Earth is 6400 km
The radius of the latitudes is \( r = R \cos \alpha \) or \( r = 6400 \cos 30^\circ \), where \( \alpha = \frac{1}{2} \theta \).

The distance between P and Q is \( |PQ| = \frac{\theta}{360^\circ} \times 2\pi r = \frac{60^\circ}{360^\circ} \times 2\pi (6400 \cos 30^\circ) \)

Therefore, \( |PQ| = \frac{60^\circ}{360^\circ} \times 2\pi (6400 \cos 30^\circ) = 5,804.91 \)

\[
\text{Time} = \frac{\text{Distance}}{\text{Speed}} = \frac{5804.9105}{600} = 9.67 \text{ hrs}, \text{ where the speed is } 600 \text{kmh}^{-1}.
\]

The points \( P(40^\circ N, 22^\circ E) \) and \( Q(40^\circ N, 48^\circ W) \) are on the earth surface. Assuming the earth is a sphere of 6,400 km and \( \pi = 3.142 \), calculate:

i. The radius of the circle of latitude through P and Q

ii. The distance PQ along the parallel of latitude.

**Solution**

The Earth Surface

Two Latitudes and Longitudes
If \( r \) and \( R \) are the radii of the small and great circles respectively, then:

\[ r = R \cos \theta , \quad R = 6,400 \text{ km} \quad \text{and} \quad \theta = 40^\circ \]

\[ r = R \cos \theta = 6400 \cos 40^\circ = 4,900 \text{ km} \]

ii. The distance of PQ

\[ \alpha = 48^\circ + 22^\circ = 70^\circ , \quad r = 4,900 \text{ km} , \quad \text{and} \quad \pi = 3.142 \]

\[ |PQ| = \frac{70^\circ}{360^\circ} \times 2 \times 3.142 \times 4,902.4 = 5,990.1881 \text{ km} \]

Activity 5.2
1. Two towns \( P(45^\circ W, 30^\circ N) \) and \( Q(25^\circ E, 30^\circ N) \) are on the surface of the earth. Find the time it will take a pilot flying at \( 500 \text{ kmh}^{-1} \) to travel from P to Q.

2. Two towns \( P(48^\circ W, 60^\circ N) \) and \( Q(22^\circ E, 60^\circ N) \) are on the surface of the earth. Find the time it will take a pilot flying at \( 700 \text{ kmh}^{-1} \) to travel from P to Q.

3. The points \( P(50^\circ N, 25^\circ E) \) and \( Q(50^\circ N, 48^\circ W) \) are on the earth surface. Assuming the earth is a sphere of \( 6,400 \text{ km} \) and \( \pi = 3.142 \), calculate:

i. the radius of the circle of latitude through P and Q

ii. the distance PQ along the parallel of latitude.

4. The points \( P(30^\circ N, 28^\circ E) \) and \( Q(30^\circ N, 52^\circ W) \) are on the earth surface. Assuming the earth is a sphere of \( 6,400 \text{ km} \) and \( \pi = 3.142 \), calculate:
i. the radius of the circle of latitude through P and Q

ii. the distance PQ along the parallel of latitude.

**Summary**

- The longitude of a place is the arc, measured in degrees along a parallel of latitude from the prime meridian.
- Since the prime meridian is arbitrary, other countries have often used their own.
- The distance represented by a degree of longitude varies upon where it is measured.
- Because the earth is not a perfect sphere, the equatorial circumference does not equal that of the meridians.
- The length of each degree is equal to the circumference divided by 360.
- Measurement along meridians of longitude accounts for the earth’s polar flattening.
- Degree lengths along meridians are not constant:
  - Mathematically, Length of a degree of longitude is \( \cos(latitude) \times 111.325 \text{ km} \).
- If \( r \) and \( R \) are the radii of the small and great circles respectively, then \( r = R \cos \theta \), and \( R = 6,400 \text{ km} \).
- The distance, \(|PQ|\) between the points, P and Q is given by \( |PQ| = \frac{\theta}{360^\circ} \times 2\pi r \).
- The time taken for a moving object on the earth surface is \( \frac{Distance}{Speed} \).
INTRODUCTION
Dear Students,
In Unit 5 Section 2, we learned that the longitude of a place is the arc, measured in degrees along a parallel of latitude from the prime meridian and measurement along meridians of longitude accounts for the earth’s polar flattening. In this section, we are going to delve into another area of mathematics called ‘Statistics’. One school of thought regards ‘Statistics’ as numerical information expressed in quantitative terms. This information may relate to objects, subjects, activities, phenomena, or regions of space.

As a matter of fact, data have no limits as to their reference, coverage, and scope. At the national and international level, these data share the contribution of agriculture, manufacturing, and services in Gross Domestic Product (GDP). At local level, individuals, firms (small or large) and institutions, produce extensive statistics on their operations. The annual reports of all institutions contain variety of data on sales, production, expenditure, inventories, capital employed, and other activities. These data are often field data, collected by employing scientific survey techniques and regularly updated.

On another hand, a student knows statistics more intimately as a subject of study just economics, mathematics, chemistry, physics, and others. To this extent, ‘Statistics’ is a discipline, which scientifically deals with data, and is often described as the science of data. In both cases, statistics has developed appropriate methods of collecting, presenting, summarizing, and analysing data.

In this section, our learning indicators are:
- scrutinize the meaning and definitions of Statistics
- outline types of data and data sources
- enumerate types of Statistics
- explain the scope of Statistics
- discuss the importance of Statistics
- summarize some limitations of statistics

**The Meaning and Definitions of Statistics**
The word ‘statistics’ is used in two senses--- plural and singular. In the plural sense, it refers to a set of figures or data. In the singular sense, statistics refers to the whole body of tools that are used to collect data, organise and interpret data and, draw conclusions from the data.

Statistics is a collection of methods for collecting, displaying, analyzing, and drawing conclusions from data.

Statistics is a very broad subject, with applications in a vast number of different fields. In generally one can say that statistics is the methodology for collecting, analyzing, interpreting and drawing conclusions from information.

It should be noted that both aspects of statistics are important if the quantitative data are to serve their purpose. If statistics, as a subject, is inadequate and consists of poor methodology, we could not know the right procedure to extract from the data and the information they contain. Similarly, if our data are defective or that they are inadequate or inaccurate, we could not reach the right conclusions even though our subject is well developed.

**Example 1**
Statistics include numerical facts and figures:
- a. The largest earthquake was measured 9.2 on the Richter scale.
- b. Men are at least 10 times more likely than women to commit murder.
- c. One in every 8 Africans has ever been infected with malaria virus.
- d. By the year 2020, there will be 15 people aged 65 and over for every new baby born.

Statistics is variously defined by authorities as follows:
1. Bowley has defined statistics as:
   (i). the science of counting,
   (ii). the science of averages
   (iii). the science of measurement of social organism regarded as a whole in all its manifestations.

2. Boddington defines ‘statistics’ as the science of estimates and probabilities.

3. King defines ‘statistics’ as the science of method of judging collective, natural or social phenomena from the results obtained by the analysis or enumeration or collection of estimates.

4. Seligman defines ‘statistics’ as a science that deals with the methods of collecting, classifying, presenting, comparing and interpreting numerical data collected to throw some light on any sphere of enquiry.
5. Spiegel defines ‘statistics’ as a scientific method for collecting, organising, summa rising, presenting and analyzing data as well as drawing valid conclusions and making reasonable decisions on the basis of such analysis.

6. Prof. Horace Secrist defines ‘statistics’ as the aggregate of facts, affected to a marked extent by multiplicity of causes, numerically expressed, enumerated or estimated according to reasonable standards of accuracy, collected in a systematic manner for a pre-determined purpose, and placed in relation to each other.

7. The science of statistics deals with the collection, analysis, interpretation, and presentation of data. We see and use data in our everyday lives.

The above definitions highlight the following major characteristics of statistics:
(i). Statistics are the aggregates of facts. It means a single figure is not statistics. For example, national income of a country for a single year is not statistics but the same for two or more years is statistics.
(ii). Statistics are affected by a number of factors. For example, sale of a product depends on a number of factors such as its price, quality, competition, and the income of the consumers.
(iii). Statistics must be reasonably accurate. Wrong figures, if analysed, will lead to erroneous conclusions. Hence, it is necessary that conclusions must be based on accurate figures.
(iv). Statistics must be collected in a systematic manner. If data are collected in a haphazard manner, they will not be reliable and will lead to misleading conclusions.
(v) Collected in a systematic manner for a pre-determined purpose
(vi). Statistics should be placed in relation to each other. If one collects data unrelated to each other, then such data will be confusing and will not lead to any logical conclusions. Data should be comparable over time and over space.

Example 2
Identify a major flaw with each interpretation in the following statistics.
a). A new advertisement for fan yoghurt and fan ice introduced last year late February resulted in a 30% increase in yoghurt sales for the months. Thus, the advertisement was effective.

A major flaw is that yoghurt consumption generally increases in the months of March, April and May regardless of advertisements. This effect is called a history effect and leads people to interpret outcomes as the result of one variable when another variable (in this case, one having to do with the passage of time) is actually responsible.

b) The more churches in a city, the more crime there is. Thus, churches lead to crime.
A major flaw is that both increased churches and increased crime rates can be explained by larger populations. In bigger cities, there are both more churches and more crime. This problem refers to the third-variable problem. Namely, a third variable can cause both situations; however, people erroneously believe that there is a causal relationship between the two primary variables rather than recognize that a third variable can cause both.

c). 75% more inter-tribal marriages are occurring this year than 25 years ago. Thus, our society accepts inter-tribal marriages.

A major flaw is that we do not have the information that we need. What is the rate at which marriages are occurring? Suppose only 1% of marriages 25 years ago were inter-tribal and so now 1.75% of marriages are inter-tribal (1.75 is 75% higher than 1). But this latter number is hardly evidence suggesting the acceptability of inter-tribal marriages.

In addition, the statistic provided does not rule out the possibility that the number of inter-tribal marriages has seen dramatic fluctuations over the years and this year is not the highest.

Again, there is simply not enough information to understand fully the impact of the statistics.

As a whole, these examples show that statistics are not only facts and figures; they are something more than that. In the broadest sense, ‘statistics’ refers to a range of techniques and procedures for analyzing, interpreting, displaying, and making decisions based on data!

Activity 5.3
1. ‘The word ‘statistics’ is used in two senses--- plural and singular’. Explain this statement.
2. There are many ways of defining statistics. Which of the definitions is best to you and why?

Types of Data and Data Sources
a). The meaning of data
1. Statistical data are the basic raw material of statistics.

2. Data may relate to an activity of our interest, a phenomenon, or a problem situation under study. They derive as a result of the process of measuring, counting and/or observing.

3. Statistical data, therefore, refer to those aspects of a problem situation that can be measured, quantified, counted, or classified.

4. Any object subject, phenomenon, or activity that generates data through this process is termed as a variable. In other words, a variable is one that shows a degree of variability when successive measurements are recorded.
5. “Data” refers to the information that has been collected from an experiment, a survey, or an historical record (i.e. “data” is plural and ‘datum’ is singular)

b). The types of data

In statistics, data are classified into two broad categories, namely quantitative data and qualitative data. This classification is based on the kind of characteristics that are measured.

Quantitative data

Quantitative data are those that can be quantified in definite units of measurement. These refer to characteristics whose successive measurements yield quantifiable observations. Depending on the nature of the variable observed for measurement, quantitative data can be further categorized as continuous and discrete data.

Quantitative data are numerical measurements that mostly arise from a natural numerical scale.

Continuous data

Continuous data represent the numerical values of a continuous variable. A continuous variable is the one that can assume any value between any two points on a line segment, thus representing an interval of values.

The values are quite precise and close to each other, yet distinguishably different. All characteristics such as time, distance, weight, length, height, thickness, velocity, temperature, tensile strength, and class scores represent continuous variables. Thus, the data recorded on these and similar other characteristics are called continuous data.

It may be noted that a continuous variable assumes the finest unit of measurement. Finest in the sense that it enables measurements to the maximum degree of precision.

Discrete data

Discrete data are the values assumed by a discrete variable. A discrete variable is the one whose outcomes are measured in fixed numbers. Such data are essentially count data.

These are derived from a process of counting, such as the number of items possessing or not possessing a certain characteristic. Common examples are the number of customers visiting a departmental store everyday, the incoming flights at an airport, and the defective items in a consignment received for sale, the number of students in a classroom and the performance rates.

Qualitative data

Qualitative data refer to qualitative characteristics of a subject or an object. A characteristic is qualitative in nature when its observations are defined and noted in terms of the presence or absence of a certain attribute in discrete numbers.
Qualitative data are measurements for which there is no natural numerical scale, but which consist of attributes, labels, or other non-numerical characteristics.

These data are further classified as nominal and rank data.

**Nominal data**

Nominal data are the outcome of classification into two or more categories of items or units comprising a sample or a population according to some quality characteristic.

Classifications of students according to sex (as males and females), of workers according to skill (as skilled, semi-skilled, and unskilled), of objects according to colours (white, black, and yellow), of employees according to the level of education (as diploma, first degree, and masters), of lecture halls according to room numbers, of jerseys according to positions, of mobile numbers, and so on.

Given any such basis of classification, it is always possible to assign each item to a particular class and make a summation of items belonging to each class. The count data so obtained are called nominal data.

**Rank data**

Rank data, on the other hand, are the result of assigning ranks to specify order in terms of the integers 1, 2, 3, ..., n. Ranks may be assigned according to the level of performance in a test (first, second, third, etc.), a contest, a competition, an interview, or a show.

The candidates appearing in an interview may be assigned ranks in integers ranging from 1 to n, depending on their performance in the interview. Ranks so assigned can be viewed as the continuous values of a variable involving performance as the quality characteristic.

**Example**

1. Identify the following measures as either quantitative or qualitative:
   i. The 30 high-temperature readings of the last 30 days.
   ii. The scores of 40 students on a Statistics test.
   iii. The blood types of 120 teachers in a basic school.
   iv. The last four digits of social security numbers of all teachers in a class.
   v. The numbers on the jerseys of 53 football players on a football team.
   vi. The genders of the first 40 newborns in the Winneba Trauma hospital in one year.
   vii. The natural hair colour of 20 randomly selected female teachers.
   viii. The ages of 20 randomly selected students.
   ix. The fuel economy in miles per gallon of 20 new cars purchased last month.
   x. The political affiliation of 500 randomly selected voters.

**Solution**
i. Quantitative.
ii. Quantitative.
iii. Quantitative.
iv. Qualitative
v. Qualitative
vi. Qualitative.
vii. Qualitative.
viii. Quantitative.
ix. Quantitative.
x. Qualitative.

2. Classify the following variables as qualitative and quantitative and as discrete or continuous:

a). Number of passengers on the MMT bus from Accra to Winneba

b). Education of a group of people in Cape Coast

C). The average weight of newborns from maternity in Kumasi

d). Altitude above sea level in Keta

e). A survey conducted with 1,015 people indicates that 40 of them are subscribers to a broadband Internet service.

f). The electronic indicates that the player radar last snapped ball 82,3mi / h

g). The time spent for a person to make a Takoradi drive to Tarkwa is approximately 2: 40h at an average speed of 100km/h

h). The students eye colour

i). Production of cashew nuts in Sunyani

j). Number of defects on TV equipment in Jomoro

k). The point obtained in each play of a given game

Solution
a. Quantitative discrete
b. qualitative
c. quantitative continuous
d. quantitative continuous
e. quantitative discrete
f. qualitative
g. quantitative continuous
h. qualitative
i. quantitative continuous
j. quantitative continuous
k. quantitative discrete

3. In a study in a school, data was collected for the following variables:

(a) age (E) time spent daily in study
(b) grade (F) distance from home to school
(c) sex (G) study site
(d) note in the discipline of Mathematics (H) number of siblings

i). The indicated variables, which are quantitative and which are qualitative?

ii) Of quantitative variables, which are continuous says.

Solution
i). a. Quantitative (A), (D), (E), (F), (H)

b. Qualitative: (B) (C) (G)

ii. Are continuous quantitative variables (E), (F) and optionally (A); the variable Age is also continuous, it can take any value in a range, although it is usually treated as discrete

c). Sources of data
Data sources are mainly of two types, namely secondary and primary. However, the third source
(internet) is a hybrid of the two sources. These can be defined as under:

Secondary data
They already exist in some form: published or unpublished -in an identifiable secondary source.
They are, generally, available from published source(s), though not necessarily in the form actually
required.

Primary data
These data do not already exist in any form, and thus have to be collected for the first time from
the primary source(s). By their very nature, these data require fresh and first-time collection
covering the whole population or a sample drawn from it.

Internet data
These data exist in the internet and can be collected for statistical analysis. By their nature, they
neither classified as primary nor secondary. Common examples are wiki, google, yahoo, Facebook
and Whatsapp.

Types of Statistics
There are two major divisions of statistics, namely descriptive statistics and inferential statistics.

Descriptive statistics
1. Descriptive statistics, also known as deductive statistics, is the branch of statistics that involves
organizing, displaying, and describing data.

2. Descriptive statistics is the branch of statistics that involves organizing, displaying, and
describing data.

3. Descriptive Statistics: deals with procedures used to summarize the information contained in a
set of measurements.

4. The term descriptive statistics deals with collecting, summarizing, and simplifying data, which
are otherwise quite unwieldy and voluminous. It seeks to achieve this in a manner that meaningful
conclusions can be readily drawn from the data.

5. Descriptive statistics is the branch of statistics concerned with describing and summarizing data,
and a set of statistics such as the mean, standard deviation, and skew that describes a distribution.

6. Descriptive statistics may be seen as comprising methods of bringing out and highlighting the
latent characteristics present in a set of numerical data. It not only facilitates an understanding of
the data and systematic reporting thereof in a manner; and also makes them amenable to further
discussion, analysis, and interpretations.
7. Descriptive statistics are numbers that are used to summarize and describe data. For example, if we are analyzing birth certificates, a descriptive statistic might be the percentage of certificates issued in Ghana, or the average age of the mothers. Any other number we choose to compute also counts as a descriptive statistic for the data from which the statistic is computed. Several descriptive statistics are often used at one time to give a full picture of the data.

8. Descriptive statistics are just descriptive. They do not involve generalizing beyond the data at hand. Generalizing from our data to another set of cases is the business of inferential statistics. Examples of descriptive statistics include measures of central tendency, dispersion, skewness, and kurtosis.

9. Organizing and summarizing data is called descriptive statistics. Two ways to summarize data are by graphing and by using numbers (e.g. finding an average).

**Inferential statistics**

1. Inferential statistics, also known as inductive statistics, goes beyond collecting, summarizing, and meaningfully presenting the related data. Instead, it consists of methods that are used for drawing inferences, or making broad generalizations, about a totality of observations on the basis of knowledge about a part of that totality.

2. Inferential statistics is the branch of statistics that involves drawing conclusions about a population based on information contained in a sample taken from that population.

3. Inferential statistics obtains a particular value from the sample information and uses it for drawing an inference about the entire population.

4. Inferential statistics helps to evaluate the risks involved in reaching inferences or generalizations about an unknown population on the basis of sample information. For example, an inspection of a sample of five battery cells drawn from a given lot may reveal that all the five cells are in perfectly good condition. This information may be used to conclude that the entire lot is good enough to buy or not.

**Scenarios of inferential statistics**

Consider a situation in which one is required to know the average body weight of all students in a given area during a certain year: Just record the weight of only 500 students, from out of a total strength of, say, 10,000, or an unknown total strength, take the average, and use this average based on incomplete weight data to represent the average body weight of all the students.
In a different situation, one may have to repeat this exercise for some future year and use the quick estimate of average body weight for a comparison. This may be needed to decide whether the weight of the students has undergone a significant change over the years compared.

Since inference is based on the examination of a sample of limited number of cells, it is equally likely that all the cells in the lot are not in order. It is also possible that all the items that may be included in the sample are unsatisfactory. This may be used to conclude that the entire lot is of unsatisfactory quality, whereas the fact may indeed be otherwise. It may, thus, be noticed that there is always a risk of an inference about a population being incorrect when based on the knowledge of a limited sample.

The rescue in such situations lies in evaluating such risks. For these and other reasons, inferential statistics provides the necessary methods. This requires an understanding of the what, why, and how of probability and probability distributions to equip ourselves with methods of drawing statistical inferences and estimating the degree of reliability of these inferences.

**Population**
A population is the complete set of observations a researcher is interested in. A population can be defined in a manner convenient for a researcher. For example, one could define a population as all girls in UEW, Winneba or the set of all girls in Level 100 students of UEW, Winneba.

A (statistical) population is the set of measurements (or record of some qualitative trait) corresponding to the entire collection of units for which inferences are to be made.

In statistics, we generally want to study a population. The population is a collection of persons, things, or objects under study.

**Census**
The desired information about a given population of our interest; may also be collected even by observing all the units comprising the population. This total coverage is called census.

Getting the desired value for the population through census is not always feasible and practical for various reasons. Apart from time and money considerations making the census operations prohibitive, observing each individual unit of the population with reference to any data characteristic may at times involve even destructive testing. In such cases, obviously, the only recourse available is to employ the partial or incomplete information gathered through a sample for the purpose.

**Sample**
A sample is a subset of a population, often taken for the purpose of statistical inference. Generally, one uses a random sample.

A sample from statistical population is the set of measurements that are actually collected in the course of an investigation.

The idea of sampling is to select a portion (or subset) of the larger population and study that portion (the sample) to gain information about the population in details. Data are the result of sampling from a population.

The totality of observations about which an inference may be drawn, or a generalization made, is called a population or a universe. The part of totality, which is observed for data collection and analysis to gain knowledge about the population, is called a sample.

Inferential statistics are computed from sample data in order to make inferences about the population. Hence, to study the population, we select a sample.

**Representative Sample**

A representative sample is a sample chosen to match the qualities of the population from which it is drawn. With a large sample size, random sampling will approximate a representative sample. Usually, the stratified random sampling is used to make a small sample more representative.

The sample must contain the characteristics of the population in order to be a representative sample. So, we must be interested in both the sample statistic and the population parameter in inferential statistics. We use the sample statistic to test the validity of the established population parameter.

In effect, a *population* is any specific collection of objects of interest. A sample is any subset or subcollection of the population, including the case that the sample consists of the whole population, in which case it is termed a census.

**Parameter**

Parameter is a value calculated in a population. For example, the mean of the numbers in a population is a parameter. Compare with a statistic, which is a value computed in a sample to estimate a parameter.

A parameter is a number that is a property of the population. Since we considered all mathematics classes to be the population, then the average number of points earned per student over all the mathematics classes is an example of a parameter.
A parameter is an unknown numerical summary of the population. A statistic is a known numerical summary of the sample which can be used to make inference about parameters.

**Statistic**
Even though the term ‘statistics’ is a field of study concerned with summarizing data, interpreting data, and making decisions based on data, there is a quantity, called ‘statistic’ calculated to estimate a value in a population.

A statistic is a number that represents a property of the sample. For example, if we consider one mathematics class to be a sample of the population of all mathematics classes, then the average number of points earned by students in that one mathematics class at the end of the semester is a statistic. The statistic is an estimate of a population parameter.

In effect, a parameter is a number that summarizes some aspect of the population as a whole, and a statistic is a number computed from the sample data.

**Example**
1. Identify each of the following data sets as either a population or a sample:
   a. The grade point averages (GPAs) of all students in a University.
   b. The GPAs of a randomly selected group of students on North Campus of UEW, Winneba.
   c. The ages of the nine Supreme Court Justices of Ghana in 1999.
   d. The gender of every second customer who enters a movie theatre.
   e. The lengths of Keta boat undertaking fishing trip in the sea.

**Solution**
   a. Population.
   b. Sample.
   c. Population.
   d. Sample.
   e. Sample.

2. A sociologist wishes to estimate the proportion of all adults in a certain region who have never married. In a random sample of 1,320 adults, 145 have never married, hence \( \frac{145}{1320} = 0.11 \) or about 11% have never married.
   a. What is the population of interest?
   b. What is the parameter of interest?
   c. What is the statistic involved?
   d. Based on this sample, do we know the proportion of all adults who have never married? Explain fully.
Solution
a. All adults in the region.
b. The proportion of the adults in the region who have never married.
c. The proportion computed from the sample, 0.1.
d. No, not exactly, but we know the approximate value of the proportion.

Example
3. Determine what the key terms refer to in the following study.
We want to know the average (mean) amount of money first-year students spend on campus that does not include school fees. We randomly survey 100 first year students at the college. Three of those students spent ₵150, ₵200, and ₵225, respectively.

Solution
a. The population is all first year students on campus.
b. The sample could be all students enrolled in one section of a beginning statistics course (although this sample may not represent the entire population).
c. The parameter is the average (mean) amount of money spent (excluding school fees) by first-year students in this term.
d. The statistic is the average (mean) amount of money spent (excluding school fees) by first-year students in the sample.
e. The data are the cedi amounts spent by the first-year students, namely ₵150, ₵200, and ₵225.

4. Determine what the key terms refer to in the following study.
A study was conducted at the North Campus of UEW, Winneba to analyze the average cumulative GPA’s of students who graduated last year. Fill in the letter of the phrase that best describes each of the items below:
1._____ Population 2._____ Statistic 3._____ Parameter 4._____ Sample 5._____ Variable 6._____ Data

a) all students who attended the college last year
b) the cumulative GPA of one student who graduated from the college last year
c) 3.65, 2.80, 1.50, 3.90
d) a group of students who graduated from the college last year, randomly selected
e) the average cumulative GPA of students who graduated from the college last year
f) all students who graduated from the college last year
g) the average cumulative GPA of students in the study who graduated from the college last year

Solution
1. f; 2. g; 3. e; 4. d; 5. b; 6. c
5. Determine what the key terms refer to in the following study.
An insurance company would like to determine the proportion of all medical doctors who have been involved in one or more malpractice lawsuits. The company selects 500 doctors at random from a professional directory and determines the number in the sample who have been involved in a malpractice lawsuit.

Solution
a. The population is all medical doctors listed in the professional directory.
b. The parameter is the proportion of medical doctors who have been involved in one or more malpractice suits in the population.
c. The sample is the 500 doctors selected at random from the professional directory.
d. The statistic is the proportion of medical doctors who have been involved in one or more malpractice suits in the sample.
e. The data are either: yes, was involved in one or more malpractice lawsuits, or no, was not.

Significance Level
In significance testing, the significance level is the highest value of a probability value for which the null hypothesis is rejected. Common significance levels are 0.05 and 0.01. If the 0.05 level is used, then the null hypothesis is rejected if the probability value is less than or equal to 0.05.

Significance Testing
Significance Testing is a statistical procedure that tests the viability of the null hypothesis. If data (or more extreme data) are very unlikely given that the null hypothesis is true, then the null hypothesis is rejected. If the data or more extreme data are not unlikely, then the null hypothesis is not rejected.

If the null hypothesis is rejected, then the result of the test is said to be significant. A statistically significant effect does not mean the effect is important.

Null Hypothesis
A null hypothesis is a hypothesis tested in significance testing. It is typically the hypothesis that a parameter is zero or that a difference between parameters is zero. For example, the null hypothesis might be that the difference between population means is zero. Experimenters typically design experiments to allow the null hypothesis to be rejected.

Alternative Hypothesis
In hypothesis testing, the null hypothesis and an alternative hypothesis are put forward. If the data are sufficiently strong to reject the null hypothesis, then the null hypothesis is rejected in favour of an alternative hypothesis. For instance, if the null hypothesis is $\mu_1 = \mu_2$ then the alternative hypothesis (for a two-tailed test) is $\mu_1 \neq \mu_2$. 
One Tailed
The last step in significance testing involves calculating the probability that a statistic would differ as much or more from the parameter specified in the null hypothesis as does the statistics obtained in the experiment.

A probability computed considering differences in only one direction, such as the statistic is larger than the parameter, is called a one-tailed probability. For example, if a parameter is 0 and the statistic is 12, a one-tailed probability (the positive tail) would be the probability of a statistic being ≥ to 12. Compare with the two-tailed probability which would be the probability of being either ≤ -12 or ≥12.

Two Tailed
The last step in significance testing involves calculating the probability that a statistic would differ as much or more from the parameter specified in the null hypothesis as does the statistics obtained in the experiment.

A probability computed considering differences in both direction (statistic either larger or smaller than the parameter) is called two-tailed probability. For example, if a parameter is 0 and the statistic is 12, a two-tailed probability would be the he probability of being either ≤ -12 or to ≥12. Compare with the one-tailed probability which would be the probability of a statistic being ≥12 if that were the direction specified in advance.

Type I Error
Type I Error in significance testing, is the error of rejecting a true null hypothesis.

Type II Error
Type II Error in significance testing, is the failure to reject a false null hypothesis.

Probability value or P-Value
In significance testing, the probability value (sometimes called the p-value) is the probability of obtaining a statistic as different or more different from the parameter specified in the null hypothesis as the statistic obtained in the experiment.

The p-value is computed assuming the null hypothesis is true. The lower the probability value, the stronger the evidence that the null hypothesis is false. Therefore, the null hypothesis is rejected if the p-value is below 0.05.

The observed significance or p-value of a specific test of hypotheses is the probability, on the supposition that H₀ is true, of obtaining a result at least as contrary to H₀ and in favour of Hₐ as the result actually observed in the sample data.
The observed significance of a test of hypotheses is the area of the tail of the distribution cut off by the test statistic (times two in the case of a two-tailed test).

**In P-Value Approach;**
- Identify the null and alternative hypotheses.
- Identify the relevant test statistic and its distribution.
- Compute from the data the value of the test statistic.
- Compute the p-value of the test.
- Compare the value computed in Step 4 to significance level \( \alpha \) and make a decision: reject \( H_0 \) if \( p \leq \alpha \) and do not reject \( H_0 \) if \( p > \alpha \).
- Formulate the decision in the context of the problem, if applicable.

**Example**
1. Make the decision in each test, based on the information provided.
   a. Testing \( H_0: \mu = 82.9 \) vs. \( H_a: \mu < 82.9 \) at \( \alpha = 0.05 \), observed significance \( p = 0.038 \).
   b. Testing \( H_0: \mu = 213.5 \) vs. \( H_a: \mu \neq 213.5 \) at \( \alpha = 0.01 \), observed significance \( p = 0.038 \).

   **Solution**
   a. reject \( H_0 \)
   b. do not reject \( H_0 \)

2. State the null and alternative hypotheses for each of the following situations (i.e., identify the correct number \( \mu_0 \) and write \( H_0: \mu = \mu_0 \) and the appropriate analogous expression for \( H_a \).)
   a. The average July temperature in a region has been 74.5°F. Perhaps it is higher now.
   b. The average weight of a female airline passenger with luggage was 145 pounds ten years ago. The FDB believes it to be higher now.
   c. The average stipend for doctoral students in a particular discipline in the University is ₵14,756. The Dean of Graduate studies believes that the national average is higher.
   d. The average room rate in hostels in a certain region is ₵82.53. A travel agent believes that the average in a particular resort area is different.
   e. The average farm size in a predominately rural state was 69.4 acres. The Minister of Agriculture asserts that it is less today.

   **Solution**
   a. \( H_0: \mu = 74.5 \) vs. \( H_a: \mu > 74.5 \)
   b. \( H_0: \mu = 145 \) vs. \( H_a: \mu > 145 \)
   c. \( H_0: \mu = 14756 \) vs. \( H_a: \mu > 14756 \)
   d. \( H_0: \mu = 82.53 \) vs. \( H_a: \mu \neq 82.53 \)
   e. \( H_0: \mu = 69.4 \) vs. \( H_a: \mu < 69.4 \)
3. Describe the two types of errors that can be made in a test of hypotheses.

**Solution**
A Type I error is made when a true $H_0$ is rejected. A Type II error is made when a false $H_0$ is not rejected.

**Activity**
1. The intention was to make a study of the number of siblings of students in the JHS three of a basic school. For this, a survey was carried out to which 60 students answered. Indicate:
   a). The study population
   b). The chosen sample; 
   c). The study variable and rate it

2. The data are the number of machines in a gym. You sample five gyms. One gym has 12 machines, one gym has 15 machines, one gym has ten machines, one gym has 22 machines, and the other gym has 20 machines. What type of data is this?

3. The data are the areas of lawns in square feet. You sample five houses. The areas of the lawns are 144 sq. feet, 160 sq. feet, 190 sq. feet, 180 sq. feet, and 210 sq. feet. What type of data is this?

**Summary**
- Statistics is a study of data: describing properties of data (descriptive statistics) and drawing conclusions about a population based on information in a sample (inferential statistics).
- Descriptive statistics make use of quantitative data (continuous and discrete), qualitative (nominal and rank) and qualitative data.
- The distinction between a population together with its parameters and a sample together with its statistics is a fundamental concept in inferential statistics.
- Information in a sample is used to make inferences about the population from which the sample was drawn.
- Inferential statistics make use of significance level, significance testing, null hypothesis, alternative hypothesis, one tailed, two tailed, type i error, type ii error and probability value or p-value.

**Scope of Statistics**
These are mainly concerned with the following:
(i) It often becomes necessary to examine how two paired data sets are related. For example, we may have data on the sales of a product and the expenditure incurred on its advertisement for a specified number of years. Given that sales and advertisement expenditure are related to each other, it is useful to examine the nature of relationship between the two and quantify the degree of that relationship. As this requires use of appropriate statistical methods, these falls under the purview of what we call regression and correlation analysis.

(ii) Situations occur quite often when we require averaging (or totalling) of data on prices and/or quantities expressed in different units of measurement. For example, price of cloth may be quoted per meter of length and that of wheat per kilogram of weight. Since ordinary methods of totalling and averaging do not apply to such price/quantity data, special techniques needed for the purpose are developed under index numbers.

(iii) Many a time, it becomes necessary to examine the past performance of an activity with a view to determining its future behaviour. For example, when engaged in the production of a commodity, monthly product sales are an important measure of evaluating performance. This requires compilation and analysis of relevant sales data over time. The more complex the activity, the more varied the data requirements. For profit maximising and future sales planning, forecast of likely sales growth rate is crucial. This needs careful collection and analysis of past sales data. All such concerns are taken care of under time series analysis.

(iv) Obtaining the most likely future estimates on any aspect(s) relating to a business or economic activity has indeed been engaging the minds of all concerned. This is particularly important when it relates to product sales and demand, which serve the necessary basis of production scheduling and planning. The regression, correlation, and time series analyses together help develop the basic methodology to do the needful. Thus, the study of methods and techniques of obtaining the likely estimates on business/economic variables comprises the scope of what we do under business forecasting.

(v) Keeping in view the importance of inferential statistics, the scope of statistics may finally be restated as consisting of statistical methods which facilitate decision-making under conditions of uncertainty. While the term statistical methods is often used to cover the subject of statistics as a whole, in particular it refers to methods by which statistical data are analysed, interpreted, and the inferences drawn for decision-making. Though generic in nature and versatile in their applications, statistical methods have come to be widely used, especially in all matters concerning business and economics.

(vi) These are also being increasingly used in biology, medicine, agriculture, psychology, and education. The scope of application of these methods has started opening and expanding in a number of social science disciplines as well. Even a political scientist finds them of increasing
relevance for examining the political behaviour and it is no surprise to find historians statistical
data, for history is essentially past data presented in certain actual format.

Example 1
Write a note on the scope and limitations of Statistics.

Importance of Statistics
There are three major functions in any business enterprise in which the statistical methods are
useful. These are as follows:
(i) The planning of operations: This may relate to either special projects or to the recurring
activities of a firm over a specified period.

(ii) The setting up of standards: This may relate to the size of employment, volume of sales,
fixation of quality norms for the manufactured product, norms for the daily output, and so forth.

(iii) The function of control: This involves comparison of actual production achieved against the
norm or target set earlier. In case the production has fallen short of the target, it gives remedial
measures so that such a deficiency does not occur again.

A point worth noting is that although these three functions—planning of operations, setting
standards, and control—are separate, but in practice they are very much interrelated.

Different authors have highlighted the importance of Statistics in business.
Croxton and Cowden give numerous uses of Statistics in business such as:
  • Project planning
  • Budgetary planning and control
  • Inventory planning and control
  • Quality control
  • Marketing, production and personnel administration.

Irwing W. Burr, dealing with the place of statistics in an industrial organisation, specifies a number
of areas where statistics is extremely useful.
  • Customer wants and market research
  • Development design and specification
  • Purchasing, production, inspection, packaging and shipping
  • Sales and complaints
  • Inventory and maintenance
  • Costs, management control, industrial engineering and research.

In the sphere of production, statistics can be useful in various ways:
• Statistical quality control methods are used to ensure the production of quality goods.
• Identifying and rejecting defective or substandard goods achieve this.
• The sale targets can be fixed on the basis of sale forecasts, which are done by using varying methods of forecasting.

Analysis of sales affected against the targets set earlier would indicate the deficiency in achievement, which may be on account of several causes:
(i) targets were too high and unrealistic
(ii) salesmen's performance has been poor
(iii) emergence of increase in competition
(iv) poor quality of company's product.

Another sphere in business where statistical methods can be used is personnel management. Here, one is concerned with the fixation of wage rates, incentive norms and performance appraisal of individual employee. The concept of productivity is very relevant here.

On the basis of measurement of productivity, the productivity bonus is awarded to the workers. Comparisons of wages and productivity are undertaken in order to ensure increases in industrial productivity.

Statistical methods could also be used to ascertain the efficacy of a certain product, say, medicine. For example, a pharmaceutical company has developed a new medicine in the treatment of bronchial asthma. Before launching it on commercial basis, it wants to ascertain the effectiveness of this medicine. It undertakes an experimentation involving the formation of two comparable groups of asthma patients.

One group is given this new medicine for a specified period and the other one is treated with the usual medicines. Records are maintained for the two groups for the specified period. This record is then analysed to ascertain if there is any significant difference in the recovery of the two groups. If the difference is really significant statistically, the new medicine is commercially launched.

Example 2
Write a note on the scope of Statistics in education.

Limitations of Statistics
Statistics has a number of limitations, pertinent among them are as follows:
(i). There are certain phenomena or concepts where statistics cannot be used. This is because these phenomena or concepts are not amenable to measurement. For example, beauty, intelligence,
courage cannot be quantified. Statistics has no place in all such cases where quantification is not possible.

(ii) Statistics reveal the average behaviour, the normal or the general trend. An application of the 'average' concept if applied to an individual or a particular situation may lead to a wrong conclusion and sometimes may be disastrous. For example, one may be misguided when told that the average depth of a river from one bank to the other is four feet, when there may be some points in between where its depth is far more than four feet. On this understanding, one may enter those points having greater depth, which may be hazardous.

(iii). Since statistics are collected for a particular purpose, such data may not be relevant or useful in other situations or cases. For example, secondary data (i.e., data originally collected by someone else) may not be useful for the other person.

(iv). Statistics are not 100 per cent precise as is Mathematics or Accountancy. Those who use statistics should be aware of this limitation.

(v). In statistical surveys, sampling is generally used as it is not physically possible to cover all the units or elements comprising the universe. The results may not be appropriate as far as the universe is concerned. Moreover, different surveys based on the same size of sample but different sample units may yield different results.

(vi). At times, association or relationship between two or more variables is studied in statistics, but such a relationship does not indicate cause and effect' relationship. It simply shows the similarity or dissimilarity in the movement of the two variables. In such cases, it is the user who has to interpret the results carefully, pointing out the type of relationship obtained.

(vii). A major limitation of statistics is that it does not reveal all pertaining to a certain phenomenon. There is some background information that statistics does not cover. Similarly, there are some other aspects related to the problem on hand, which are also not covered. The user of Statistics has to be well informed and should interpret Statistics keeping in mind all other aspects having relevance on the given problem.

Example 3
Write a note on the use of Statistics in education.

Misuses of statistics
Apart from the limitations of statistics mentioned above, there are misuses of it. Many people, knowingly or unknowingly, use statistical data in wrong manner. Let us see what the main misuses of statistics are so that the same could be avoided when one has to use statistical data.
The misuse of Statistics may take several forms some of which are explained below.

(i). Sources of data not given: At times, the source of data is not given. In the absence of the source, the reader does not know how far the data are reliable. Further, if he wants to refer to the original source, he is unable to do so.

(ii). Defective data: Another misuse is that sometimes one gives defective data. This may be done knowingly in order to defend one's position or to prove a particular point. This apart, the definition used to denote a certain phenomenon may be defective. For example, in case of data relating to unemployed persons, the definition may include even those who are employed, though partially. The question here is how far it is justified to include partially employed persons amongst unemployed ones.

(iii). Unrepresentative sample: In statistics, several times one has to conduct a survey, which necessitates to choose a sample from the given population or universe. The sample may turn out to be unrepresentative of the universe. One may choose a sample just on the basis of convenience. He may collect the desired information from either his friends or nearby respondents in his neighbourhood even though such respondents do not constitute a representative sample.

(iv). Inadequate sample: Earlier, we have seen that a sample that is unrepresentative of the universe is a major misuse of statistics. This apart, at times one may conduct a survey based on an extremely inadequate sample. For example, in a city we may find that there are 1,00,000 households. When we have to conduct a household survey, we may take a sample of merely 100 households comprising only 0.1 per cent of the universe. A survey based on such a small sample may not yield right information.

(v). Unfair Comparisons: An important misuse of statistics is making unfair comparisons from the data collected. One way of unfair comparison is to construct an index of production choosing the base year where the production was much less. Then he may compare the subsequent year's production from this low base. Such a comparison will undoubtedly give a rosy picture of the production though in reality it is not so.

Again, a source of unfair comparisons could be when one makes absolute comparisons instead of relative ones. An absolute comparison of two figures, say, of production or export, may show a good increase, but in relative terms it may turnout to be very negligible.

Also, an unfair comparison is when the population in two cities is different, but a comparison of overall death rates and deaths by a particular disease is attempted. Such a comparison is wrong.
In addition, when data are not properly classified or when changes in the composition of population in the two years are not taken into consideration, comparisons of such data would be unfair as they would lead to misleading conclusions.

(vi). Unwanted conclusions: Another misuse of statistics may be on account of unwarranted conclusions. This may be as a result of making false assumptions. For example, while making projections of population in the next five years, one may assume a lower rate of growth though the past two years indicate otherwise.

Again, sometimes one may not be sure about the changes in business environment in the near future. In such a case, one may use an assumption that may turn out to be wrong.

In addition, source of unwarranted conclusion may be the use of wrong average. Suppose in a series there are extreme values, one is too high while the other is too low, such as 800 and 50.

Lastly, the use of an arithmetic average in such a case may give a wrong idea. Instead, harmonic mean would be proper in such a case.

(vii). Confusion of correlation and causation: In statistics, several times one has to examine the relationship between two variables. A close relationship between the two variables may not establish a cause-and-effect-relationship in the sense that one variable is the cause and the other is the effect. It should be taken as something that measures degree of association rather than try to find out causal relationship.

Example 4
What are the major misuses of Statistics in Ghana?

Activity 4.1
1. Explain the importance of statistics to education, trade, commerce and business.

2. What are the major limitations of Statistics? Explain with suitable examples.

Summary
We have learned that:
- The scope of statistics are pairing two data sets, averaging (or totalling) of data, examining the past performance of an activity with a view to determining its future behaviour, obtaining the most likely future estimates on any aspect(s), keeping in view the importance
of inferential statistics, facilitating decision-making under conditions of uncertainty, and being increasingly used in biology, medicine, agriculture, psychology and education.

- There are three major functions of statistics, namely planning of operations, setting up of standards, and function of control.

- Limitations of statistics lies in certain phenomena or concepts where statistics cannot be used, the average, the normal or the general trend of a behaviour, data may not be relevant or useful in other situations or cases, and statistics are not 100 per cent precise.

- The other limitations are sampling is generally used as it is not physically possible to cover all the units or elements comprising the universe, association or relationship between two or more variables does not indicate cause and effect' relationship, and statistics does not reveal all pertaining to a certain phenomenon.

- The misuse of statistics are sources of data not given, defective data, unrepresentative sample, inadequate sample, unwanted conclusions, and confusion of correlation and causation.

UNIT 5 SECTION 4 TEACHING MEASURES OF CENTRAL TENDENCIES AND LOCATIONS

INTRODUCTION
In the previous section we saw that there are two ways to describe data. This section defines the three most common measures of central tendency, namely the mean, the median, and the mode. The relationships among these measures of central tendency are paramount to this section.
This section also gives the basic definitions of the mean, median and mode. A further discussion of the relative merits and proper applications of these statistics is presented. Another way to summarize the data in a quantitative variable, plus tables and graphs, is to present them in the form of the numeric values, called measures of central tendency. These measures, when calculated from population data, are called parameters and, when calculated from sample data are called estimators or statistics.

**Learning indicators**

1. Identify situations in which knowing the centre of a distribution would be valuable.
2. Give different ways the centre of a distribution can be defined.
3. Recognize, describe, and calculate the measures of the centre of data.
4. Recognize, describe, and calculate the measures of location of data: quartiles and percentiles.
5. The present lesson imparts understanding of the calculations and main properties of measures of central tendency, including mean, mode, median, quartiles, and percentiles.

**Situations Required for the Central Tendency**

As to whether statistical data is quite elaborate or quite brief depends on the nature of the data and the purpose for which the same data have been collected. While describing data statistically or verbally, one must ensure that the description is neither too brief nor too lengthy.

The measures of central tendency are required to enable us to compare two or more distributions pertaining to the same time period or within the same distribution over time. For example, we
require the central tendency to measure the average consumption of rice in two different regions at the same period or in one region at two different years.

The ‘centre’ of a data set is a way of describing location of the data. The three most widely used measures of the ‘centre’ of the data are the mean (average), the mode and the median. To calculate the mean weight of 50 people, add the 50 weights together and divide by 50. To find the mode of 50 people’s weight, locate the entry with the highest occurrence of the weight. To find the median weight of the 50 people, order the data and find the number that splits the data into two equal parts.

The centre of a distribution could be defined three ways; namely the point on which a distribution would balance, the value whose average absolute deviation from all the other values is minimized, and the value whose squared difference from all the other values is minimized. The mean is the point on which a distribution would balance, the median is the value that minimizes the sum of absolute deviations, and the mean is the value that minimizes the sum of the squared deviations.

The median is generally a better measure of the centre when there are extreme values or outliers because it is not affected by the precise numerical values of the outliers. The mean is the most common measure of the centre. Any of three are required to measure the central location of the data.

Generally, descriptive measures are measures of position, of central tendency, of dispersion, and of skewness and kurtosis. While describing data statistically or verbally, one requires the central tendency to describe the data.

In everyday life, the word “average” is used in a variety of ways – batting averages, average life expectancies, and so on but the meaning is similar, usually the centre of a distribution. In the mathematical world, where everything must be precise, we require the central tendency to define several ways of finding the centre of a set of data.

**THE ARITHMETIC MEAN**

**Introduction**
The words “mean” and “average” are often used interchangeably. The substitution of one word for the other is a common practice. The technical term is “arithmetic mean” and “average” is technically a centre location. However, in practice among non-statisticians, “average” is commonly accepted for “arithmetic mean”.

The first measure of central location is the usual “average” that is familiar to everyone. In the formula, we normally introduce the standard summation notation Σ, where Σ is the capital Greek
letter sigma. In general, the notation $\Sigma$ followed by a second mathematical symbol means to add up all the values that the second symbol can take in the context of the problem.

**Various conceptions of the mean**

In definition of the mean, we follow the convention of using lowercase $n$ to denote the number of measurements in a sample, which is called the sample size. The sample mean of a set of $n$ sample data is the numbers— defined by the formula.

The mean is the sum of all the values in a set, divided by the number of values. The mean of a whole population is usually denoted by $\mu$, while the mean of a sample is usually denoted by $\bar{x}$. The mean is sensitive to any change in value, unlike the median and mode, where a change to an extreme (in the case of a median) or uncommon (in the case of a mode) value usually has no effect.

The mean is mostly used measure of central tendency, and is defined as the sum of the expected values of all observations (observation is an element of a sample) divided by the number of observations.

The Greek letter $\mu$ (pronounced ‘mew’) represents the *population mean*. One of the requirements for the *sample mean* to be a good estimate of the *population mean* is for the sample taken to be truly random.

The arithmetic mean is the most common measure of central tendency. It is simply the sum of the numbers divided by the number of numbers. The symbol “M” is used for the mean of a sample, the formula for $\mu$ is $\mu = \frac{\sum x}{N}$, where $\sum X$ is the sum of all the numbers in the population and $N$ is the number of numbers in the population.

Note that although the arithmetic mean is not the only “mean”. We have the geometric mean, the harmonic mean, the trimean and the trimmed mean just to mention a few. However, it is by far the most commonly used mean. However, if the term “mean” is used without specifying, it is assumed to refer to the arithmetic mean.

**Strategies of Computing the Arithmetic Mean**

**Ungrouped data**

In case of ungrouped data where weights are involved, our approach for calculating arithmetic mean will be different from the one used earlier. The commonest approach is to add all the observations and dividing the sum by the number of observations. For instance, we have the observations: 10, 15, 30, 7, 42, 79 and 83. These are seven observations.
Symbolically, the arithmetic mean, also called simply *mean* is \( \bar{x} = \frac{\sum x}{n} \), where \( x \) is the data set, \( n \) is the sample size and \( \bar{x} \) is simple mean.

**Examples 1**

1. TB data indicating the number of months a patient with TB lives after taking a new antibody drug are as follows (smallest to largest): 3; 4; 8; 8; 10; 11; 12; 13; 14; 15; 15; 16; 16; 17; 17; 18; 21; 22; 22; 24; 24; 25; 26; 27; 27; 29; 31; 32; 33; 33; 34; 34; 35; 37; 40; 44; 44; 47; calculate the mean.

**Solution**

\[
\bar{x} = \frac{\sum x}{n} = \frac{3 + 4 + 8 + \ldots + 44 + 47}{40} = 23.6
\]

2. The number of touchdown (TD) passes thrown by each of the 31 teams in the National Football League in the 2010 season was shown below: 37, 33, 33, 32, 29, 28, 28, 23, 22, 22, 22, 21, 21, 21, 21, 21, 20, 20, 19, 19, 18, 18, 18, 18, 16, 15, 14, 14, 14, 12, 12, 9, 6. Calculate the mean number of touchdown passes thrown.

**Solution**

\[
\bar{x} = \frac{\sum x}{n} = \frac{37 + 37 + 33 + \ldots + 9 + 6}{31} = 20.4516
\]

3. A random sample of ten students was taken from the student body of a class and their GPAs were recorded as follows: 1.90, 3.00, 2.53, 3.71, 2.12, 1.76, 2.71, 1.39, 4.00, 3.33. Find the sample mean.

**Solution**

\[
\bar{x} = \frac{\sum x}{n} = \frac{1.90 + 3.00 + 2.53 + \ldots + 4.00 + 3.33}{10} = 2.65
\]

**Discrete frequency table or weighted arithmetic mean**

When each value in the data set is not unique, the mean can be calculated by multiplying each distinct value by its frequency and then dividing the sum by the total number of data values.

Symbolically, the *mean* is \( \bar{x} = \sum fx / n \), where \( x \) is the data set, \( f \) is the frequency and \( \bar{x} \) is simple mean.
Example
1. A random sample of 19 women beyond child-bearing age gave the following data, where \( x \) is the number of children and \( f \) is the frequency of that value, the number of times it occurred in the data set. Use the table to find the sample mean.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>3</td>
</tr>
<tr>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
</tr>
</tbody>
</table>

Solution
- The data are presented by means of a data frequency table.
- Each number in the first line of the table is a number that appears in the data set.
- Each number below it is how many times it occurs. Thus the value 0 is observed three times, that is, three of the measurements in the data set are 0, the value 1 is observed six times, and so on.
- In the context of the problem, three women in the sample have had no children, six have had exactly one child, and so on.
- The explicit list of all the observations in this data set is therefore 0,0,0,1,1,1,1,1,2,2,2,2,2,3,3,3,4.
- The sample mean could have been \( \bar{x} = \sum \frac{x}{n} = \frac{0 + 0 + 0 + 1 + \ldots + 3 + 3 + 4}{19} = 1.6316 \)

However, because the sample size comes directly from a table, without first listing the entire data set, the sum of the frequencies \( n = 3 + 6 + 3 + 1 = 19 \).

The sample mean can be computed directly from the table as well:
\[
\bar{x} = \sum \frac{fx}{n} = \frac{0\times3 + 1\times6 + 2\times6 + 3\times3 + 4\times1}{19} = 1.6316
\]

2. Calculate the average height of infants according to the table below.

<table>
<thead>
<tr>
<th>Height (cm)</th>
<th>Number of infants</th>
</tr>
</thead>
<tbody>
<tr>
<td>50-54</td>
<td>4</td>
</tr>
<tr>
<td>54-58</td>
<td>9</td>
</tr>
<tr>
<td>58-62</td>
<td>11</td>
</tr>
<tr>
<td>62-66</td>
<td>8</td>
</tr>
<tr>
<td>66-70</td>
<td>5</td>
</tr>
<tr>
<td>70-74</td>
<td>2</td>
</tr>
</tbody>
</table>
### Solution

<table>
<thead>
<tr>
<th>Height (cm)</th>
<th>Number (f)</th>
<th>Class Mark (x)</th>
<th>fx</th>
</tr>
</thead>
<tbody>
<tr>
<td>50-54</td>
<td>4</td>
<td>52</td>
<td>208</td>
</tr>
<tr>
<td>54-58</td>
<td>9</td>
<td>56</td>
<td>504</td>
</tr>
<tr>
<td>58-62</td>
<td>11</td>
<td>60</td>
<td>660</td>
</tr>
<tr>
<td>62-66</td>
<td>8</td>
<td>64</td>
<td>512</td>
</tr>
<tr>
<td>66-70</td>
<td>5</td>
<td>68</td>
<td>340</td>
</tr>
<tr>
<td>70-74</td>
<td>2</td>
<td>72</td>
<td>216</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>40</strong></td>
<td></td>
<td><strong>2440</strong></td>
</tr>
</tbody>
</table>

\[
\bar{x} = \frac{\sum fx}{n} = \frac{2440}{40} = 61
\]

3. Suppose Aku has secured Mid-term test (30), Laboratory (25) and Final exam (20) in three tests. Assuming that the weights assigned to the three tests are 2 points, 3 points and 5 points in Mid-term test, Laboratory and Final exam respectively, calculate her mean score.

### Solution

<table>
<thead>
<tr>
<th>Type of test</th>
<th>Relative (w) weight</th>
<th>Mark</th>
<th>wx</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mid term</td>
<td>2</td>
<td>30</td>
<td>60</td>
</tr>
<tr>
<td>Laboratory</td>
<td>3</td>
<td>25</td>
<td>75</td>
</tr>
<tr>
<td>Final exam</td>
<td>5</td>
<td>20</td>
<td>100</td>
</tr>
<tr>
<td><strong>Totals</strong></td>
<td><strong>10</strong></td>
<td></td>
<td><strong>235</strong></td>
</tr>
</tbody>
</table>

\[
\bar{x} = \frac{\sum xw}{n} = \frac{60 + 75 + 100}{2 + 3 + 5} = 23.5.
\]

4. An investor is fond of investing in equity shares. During a period of falling prices in the stock exchange, a stock is sold at ₦120 per share on one day, ₦105 on the next and ₦90 on the third day. The investor has purchased 50 shares on the first day, 80 shares on the second day and 100 shares on the third day. What average price per share did the investor pay?

### Solution

<table>
<thead>
<tr>
<th>Day</th>
<th>Price per share (x)</th>
<th>Number of shares (w)</th>
<th>Amount paid (wx)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>120</td>
<td>50</td>
<td>6000</td>
</tr>
<tr>
<td>2</td>
<td>105</td>
<td>80</td>
<td>8400</td>
</tr>
<tr>
<td>3</td>
<td>90</td>
<td>100</td>
<td>9000</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td></td>
<td><strong>230</strong></td>
<td><strong>23,400</strong></td>
</tr>
</tbody>
</table>
\[
\overline{x} = \frac{\sum xw}{n} = \frac{6000 + 8400 + 9000}{50 + 80 + 100} = 101.70
\]

5. The mean of the following frequency distribution was found to be 1.46. Use the table to calculate the missing frequencies.

<table>
<thead>
<tr>
<th>Number of accidents</th>
<th>Number of days</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>46</td>
</tr>
<tr>
<td>1</td>
<td>X</td>
</tr>
<tr>
<td>2</td>
<td>Y</td>
</tr>
<tr>
<td>3</td>
<td>25</td>
</tr>
<tr>
<td>4</td>
<td>10</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Total</td>
<td>200</td>
</tr>
</tbody>
</table>

**Solution**

Here we are given the total number of frequencies and the arithmetic mean. We have to determine the two frequencies that are missing. Let us assume that the frequency against 1 accident is \( x \) and against 2 accidents is \( y \). If we can establish two simultaneous equations, then we can easily find the values of \( x \) and \( y \).

\[
\overline{x} = \frac{\sum fx}{n} = \frac{0.46(1.x) + (2.y) + (3.25) + (4.10) + (5.5)}{200} = 1.46 = \frac{(1.x) + (2.y) + 140}{200}
\]

\[1.46(200) = 1.x + 2.y + 140 \text{ or } 1.46(200) = x + 2.y = 152 \]……………..(1)

\[x + y = 200 - (46 + 25 + 10 + 5) \text{ or } x + y = 114 \]………………………..(2)

Now subtracting equation (ii) from equation (i), we get \( y = 38 \)

Substituting the value of \( y = 38 \) in equation (2), we get \( x = 76 \)

Hence, the missing frequencies are:

Against accident 1: 76
Against accident 2: 38

**Grouped Frequency Table**

When only grouped data is available, you do not know the individual data values (we only know intervals and interval frequencies); therefore, you cannot compute an exact mean for the data set. What we must do is estimate the actual mean by calculating the mean of a frequency table.
A frequency table is a data representation in which grouped data is displayed along with the corresponding frequencies.

To calculate the mean from a grouped frequency table we can apply the basic definition of mean: \( \text{mean} = \frac{\text{data sum}}{\text{number of data values}} \). We simply need to modify the definition to fit within the restrictions of a frequency table.

Since we do not know the individual data values we can instead find the midpoint of each interval. The midpoint is \( \bar{x} = \frac{\text{lower boundary} + \text{upper boundary}}{2} \).

We can now modify the mean definition to be \( \text{Mean of Frequency Table} \) is \( \bar{x} = \frac{\sum fx}{\sum f} \), where \( f \) is the frequency of the interval and \( x \) is the midpoint of the interval.

When each value in the data set is not unique to the interval data, the mean can be calculated by finding the midpoints of the class intervals, multiplying each distinct frequency by its midpoint and then dividing the sum of the products by the total number of data frequencies (\textit{Stats3}).

Symbolically, the \textit{mean} is \( \bar{x} = \frac{\sum fx}{\sum f} \), where \( x \) is the data set, \( f \) is the frequency, \( \sum f \) is the sum of the frequencies and \( \bar{x} \) is simple mean. Note that \( \sum f \) is the same as \( n \).

\textbf{Strategies of computing the mean of grouped data}

The strategies of computing the arithmetic mean from a grouped data are direct method, short-cut or assumed method and step-deviation or coded method.

\textbf{(i) Direct method}

In the case of direct method, the formula \( \bar{x} = \frac{\sum fx}{n} = \frac{\sum fx}{\sum f} \) is used. Here, \( x \) is mid-point of various classes, \( f \) is the frequency of each class and \( n \) is the total number of frequencies.

\textbf{Example}

1. The following is a frequency table displaying Professor Adu students’ statistic test scores. Use the table to find the best estimate of the class mean.

<table>
<thead>
<tr>
<th>Grade interval</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>50–56.5</td>
<td>1</td>
</tr>
<tr>
<td>56.5–62.5</td>
<td>0</td>
</tr>
<tr>
<td>62.5–68.5</td>
<td>4</td>
</tr>
<tr>
<td>68.5–74.5</td>
<td>4</td>
</tr>
<tr>
<td>74.5–80.5</td>
<td>2</td>
</tr>
<tr>
<td>80.5–86.5</td>
<td>3</td>
</tr>
</tbody>
</table>
Solution

• Find the midpoints for all intervals
• Calculate the sum of the product of each interval frequency and midpoint.

<table>
<thead>
<tr>
<th>Grade interval</th>
<th>Midpoint (x)</th>
<th>Number of students (f)</th>
<th>fx</th>
</tr>
</thead>
<tbody>
<tr>
<td>50–56.5</td>
<td>53.25</td>
<td>1</td>
<td>53.25</td>
</tr>
<tr>
<td>56.5–62.5</td>
<td>59.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>62.5–68.5</td>
<td>65.5</td>
<td>4</td>
<td>262</td>
</tr>
<tr>
<td>68.5–74.5</td>
<td>71.5</td>
<td>4</td>
<td>286</td>
</tr>
<tr>
<td>74.5–80.5</td>
<td>77.5</td>
<td>2</td>
<td>155</td>
</tr>
<tr>
<td>80.5–86.5</td>
<td>83.5</td>
<td>3</td>
<td>250.50</td>
</tr>
<tr>
<td>86.5–92.5</td>
<td>89.5</td>
<td>4</td>
<td>358</td>
</tr>
<tr>
<td>92.5–98.5</td>
<td>95.5</td>
<td>1</td>
<td>95.5</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>--</strong></td>
<td><strong>19</strong></td>
<td><strong>1460.25</strong></td>
</tr>
</tbody>
</table>

• Calculate the mean by dividing the sum of the product of each interval frequency by the midpoint: \( \bar{x} = \frac{\sum fx}{\sum f} = \frac{1460.25}{19} = 76.86 \).

2. The following table gives the marks of 58 students in Statistics. Calculate the average marks of this group.

<table>
<thead>
<tr>
<th>Marks</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-10</td>
<td>4</td>
</tr>
<tr>
<td>10-20</td>
<td>8</td>
</tr>
<tr>
<td>20-30</td>
<td>11</td>
</tr>
<tr>
<td>30-40</td>
<td>15</td>
</tr>
<tr>
<td>40-50</td>
<td>12</td>
</tr>
<tr>
<td>50-60</td>
<td>6</td>
</tr>
<tr>
<td>60-70</td>
<td>2</td>
</tr>
</tbody>
</table>
Solution

<table>
<thead>
<tr>
<th>Marks</th>
<th>Midpoint (x)</th>
<th>Number of students (f)</th>
<th>fx</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-10</td>
<td>5</td>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>10-20</td>
<td>15</td>
<td>8</td>
<td>120</td>
</tr>
<tr>
<td>20-30</td>
<td>25</td>
<td>11</td>
<td>275</td>
</tr>
<tr>
<td>30-40</td>
<td>35</td>
<td>15</td>
<td>525</td>
</tr>
<tr>
<td>40-50</td>
<td>45</td>
<td>12</td>
<td>540</td>
</tr>
<tr>
<td>50-60</td>
<td>55</td>
<td>6</td>
<td>330</td>
</tr>
<tr>
<td>60-70</td>
<td>65</td>
<td>2</td>
<td>130</td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td>58</td>
<td>1940</td>
</tr>
</tbody>
</table>

\[ \bar{x} = \frac{\sum fx}{\sum f} = \frac{1940}{58} = 33.45 \] marks or 33 marks approximately.

It should be noted that the mid-point of each class is taken as a good approximation of the true mean of the class. This is based on the assumption that the values are distributed fairly evenly throughout the interval. When large numbers of frequency occur, this assumption is usually accepted.

(ii) Short-cut or assumed method

In the case of short-cut method, the concept of arbitrary mean is followed. The formula for calculation of the arithmetic mean by the short-cut method is given below:

\[ \bar{x} = A + \frac{\sum fd}{n} = A + \frac{\sum fd}{\sum f}, \] where \( A \) is the arbitrary or assumed mean, \( f \) is the frequency, and \( d \) is the deviation from the arbitrary or assumed mean.

When the values are extremely large and/or in fractions, the use of the direct method would be very cumbersome. In such cases, the short-cut method is preferable. This is because the calculation work in the short-cut method is considerably reduced particularly for calculation of the product of values and their respective frequencies.

However, when calculations are not made manually but by a machine calculator, it may not be necessary to resort to the short-cut method, as the use of the direct method may not pose any problem.

The second term in the formula \( \left( \sum fd \div n \right) \) is the correction factor for the difference between the actual mean and the assumed mean.
If the assumed mean turns out to be equal to the actual mean, \( \sum fd \div n \) will be zero. The use of the short-cut method is based on the principle that the total of deviations taken from an actual mean is equal to zero.

As such, the deviations taken from any other figure will depend on how the assumed mean is related to the actual mean. While one may choose any value as assumed mean, it would be proper to avoid extreme values, that is, too small or too high to simplify calculations.

Note that a value apparently close to the arithmetic mean and either usually come from the class with the highest frequency or class in the middle should be chosen.

**Example**
The following table gives the marks of 58 students in Statistics. Calculate the average mark of this group.

<table>
<thead>
<tr>
<th>Marks</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-10</td>
<td>4</td>
</tr>
<tr>
<td>10-20</td>
<td>8</td>
</tr>
<tr>
<td>20-30</td>
<td>11</td>
</tr>
<tr>
<td>30-40</td>
<td>15</td>
</tr>
<tr>
<td>40-50</td>
<td>12</td>
</tr>
<tr>
<td>50-60</td>
<td>6</td>
</tr>
<tr>
<td>60-70</td>
<td>2</td>
</tr>
</tbody>
</table>

**Solution**

<table>
<thead>
<tr>
<th>Marks</th>
<th>Midpoint (x)</th>
<th>Number (f)</th>
<th>Deviations (d=35)</th>
<th>fd</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-10</td>
<td>5</td>
<td>4</td>
<td>-30</td>
<td>-120</td>
</tr>
<tr>
<td>10-20</td>
<td>15</td>
<td>8</td>
<td>-20</td>
<td>-160</td>
</tr>
<tr>
<td>20-30</td>
<td>25</td>
<td>11</td>
<td>-10</td>
<td>-110</td>
</tr>
<tr>
<td>30-40</td>
<td>35</td>
<td>15</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>40-50</td>
<td>45</td>
<td>12</td>
<td>10</td>
<td>120</td>
</tr>
<tr>
<td>50-60</td>
<td>55</td>
<td>6</td>
<td>20</td>
<td>120</td>
</tr>
</tbody>
</table>
\[
\bar{x} = A + \frac{\sum fd}{n} = 35 + \frac{-90}{58} = 33.45 \text{ or 33 marks approximately.}
\]

(iii) **Step-deviation or coded method**

In the case of step-deviation, the concepts of arbitrary mean and coded value are followed. The formula for calculation of the arithmetic mean by the step-deviation method is given below:

\[
\bar{x} = A + \frac{\sum fu}{n} \times C = A + \frac{\sum fu}{\sum f} \times C,
\]

where \(A\) is the arbitrary or assumed mean, \(f\) is the frequency, \(u\) is the quotient from the assumed mean and \(C\) is the class interval of the ‘\(A\)’ class. The term \((u = d / C)\) is the correction factor for the deviations and the assumed mean.

If the class intervals turn out to be equal, then \((C)\) will be equal. The use of the step-deviation method is based on the principle that the interval intervals are equal and that any other class chosen will not affect the product \((fu)\).

**Example**

The following table gives the marks of 58 students in Statistics. Calculate the average mark of this group.

<table>
<thead>
<tr>
<th>Marks</th>
<th>Number of students</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-10</td>
<td>4</td>
</tr>
<tr>
<td>10-20</td>
<td>8</td>
</tr>
<tr>
<td>20-30</td>
<td>11</td>
</tr>
<tr>
<td>30-40</td>
<td>15</td>
</tr>
<tr>
<td>40-50</td>
<td>12</td>
</tr>
<tr>
<td>50-60</td>
<td>6</td>
</tr>
<tr>
<td>60-70</td>
<td>2</td>
</tr>
</tbody>
</table>

**Solution**

<table>
<thead>
<tr>
<th>Marks</th>
<th>Midpoint (x)</th>
<th>Number (f)</th>
<th>Deviations (d=35)</th>
<th>(u=d/c)</th>
<th>(fu)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-10</td>
<td>5</td>
<td>4</td>
<td>-30</td>
<td>-3</td>
<td>-12</td>
</tr>
<tr>
<td>10-20</td>
<td>15</td>
<td>8</td>
<td>-20</td>
<td>-2</td>
<td>-16</td>
</tr>
<tr>
<td>20-30</td>
<td>25</td>
<td>11</td>
<td>-10</td>
<td>-1</td>
<td>-11</td>
</tr>
<tr>
<td>30-40</td>
<td>35</td>
<td>15</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>40-50</td>
<td>45</td>
<td>12</td>
<td>10</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>50-60</td>
<td>55</td>
<td>6</td>
<td>20</td>
<td>2</td>
<td>12</td>
</tr>
</tbody>
</table>
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
60-70 & 65 & 2 & 30 & 3 & 6 \\
\hline
\text{Total} & & 58 & & & -9 \\
\hline
\end{array}
\]

\[
\bar{x} = A + \frac{\sum f u}{n} \times c = 35 + \frac{-9}{58} \times 10 = 33.45 \text{ or 33 marks approximately.}
\]

**Comparisons of the three strategies**

It will be seen that the answer in each of the three cases is the same. However, the step-deviation method is the most convenient on account of simplified calculations.

It may also be noted that if we select a different arbitrary assumed mean and recalculate deviations, we should get the same answer.

Again, the arithmetic mean can be calculated by using any of the three different methods.

**Some important characteristics of the arithmetic mean**

1. The sum of the deviations of the individual items from the arithmetic mean is always zero. This means \( \sum (x - \bar{x}) = 0 \), where \( x \) is the value of an item and \( \bar{x} \) is the arithmetic mean.

Since the sum of the deviations in the positive direction is equal to the sum of the deviations in the negative direction, the arithmetic mean is regarded as a measure of central tendency.

2. The sum of the squared deviations of the individual items from the arithmetic mean is always minimum. In other words, the sum of the squared deviations taken from any value other than the arithmetic mean will be higher.

3. As the arithmetic mean is based on all the items in a series, a change in the value of any item will lead to a change in the value of the arithmetic mean.

4. In the case of highly skewed distribution, the arithmetic mean may get distorted on account of a few items with extreme values. In such a case, it may cease to be the representative characteristic of the distribution.

**Advantages of the arithmetic mean**

1. It is more representative because it is based on all the observations.

2. It is capable of being used for further estimations.

3. Its value is always finite and has no doubt.

4. It is the most widely used and popular average.
5. It is least affected by fluctuations of samplings

6. It is useful for calculating totals and combining different means.

7. It is easy to calculate and simple to understand.

8. Allows combining with results from other similar groups

9. It can be computed even if the detailed distribution is not known but some of the observations and number of the observations are known.

**Disadvantages of the mean**

1. One disadvantage of the mean is that a small number of extreme values can distort its value. For example, the mean of the set \{1, 1, 1, 2, 2, 3, 3, 3, 200\} is 24, even though almost all of the members were very small.

A variation called the trimmed mean, where the smallest and largest quarters of the values are removed before the mean is taken, can solve this problem.

2. The mean value may not exist among the observations.

3. The interpretations of the mean may be absurd if it is decimal.

4. It can give fallacy conclusions of different data sets.

5. The mean value is meaningless in qualitative data.

6. The mean cannot be estimated from any graph.

7. The mean cannot be computed when class intervals have open ends

**Geometric Mean**

The geometric mean is computed by multiplying all the numbers together and then taking the nth root of the product. For example, for the numbers 1, 10, and 100, the product of all the numbers is 

$$1 \times 10 \times 100 = 1000.$$ 

Since there are three numbers, we take the cubed root of the product (1,000) which is equal to 10. The formula for the geometric mean is therefore

$$Geometric\ mean = \left( \prod X \right)^{\frac{1}{n}},$$

where the symbol $\prod$ means to multiply.

Therefore, the equation tells us to multiply all the values of $X$ and then raise the result to the $1/n$th power. Raising a value to the $1/n$th power is the same as taking the nth root of the value. In this case, $1000^{1/3}$ is the cube root of 1,000.
The geometric mean has a close relationship with logarithms. The arithmetic mean of three logs is 1. The anti-log of this arithmetic mean of 1 is the geometric mean. The anti-log of 1 is \(10^1 = 10\).

Again, the geometric mean is a measure of central tendency. The geometric mean of \(n\) numbers is obtained by multiplying all of them together, and then taking the \(n\)th root of them. For instance, for the numbers 1, 10, and 100, the product of all the numbers is \(1 \times 10 \times 100 = 1000\). Since there are three numbers, we take the cubed root of the product (1,000) which is 10.

Alternatively, geometric mean is defined at the \(n\)th root of the product of \(n\) observations of a distribution. Symbolically, \(GM = \sqrt[n]{x_1 \times x_2 \times x_3 \times \ldots \times x_n}\)

If we have only two observations, \(x_1\) and \(x_2\) then \(GM = \sqrt{x_1 \times x_2}\). Similarly, if there are only three observations, then we have to calculate the cube root of the product of these three observations such as \(GM = \sqrt[3]{x_1 \times x_2 \times x_3}\); and so on.

When the number of items is large, it becomes extremely difficult to multiply the numbers and to calculate the root. To simplify the calculations, logarithms are used. That is, \(\log GM = \frac{\sum \log x}{n}\)

and \(GM = \text{anti} \log \left( \frac{\sum \log x}{n} \right)\).

Note that the geometric mean only makes sense if all the numbers are positive. The geometric mean is an appropriate measure to use for averaging rates.

The geometric mean is most suitable in the following three cases:
1. Averaging rates of change.
2. The compound interest formula.
3. Discounting and capitalization.

**Example**
1. Find out the geometric mean of 2, 4 and 8.

**Solution**

\[
\log GM = \frac{\sum \log x}{n} = \frac{\log 2 + \log 4 + \log 8}{3}
\]

\[
\log 2 + \log 4 + \log 8 = 0.3010 + 0.6021 + 0.9031 = 1.8062
\]

\[
\frac{1.8062}{3} = 0.60206
\]

\(GM = \text{anti} \log (0.60206) = 4\)
2. Bayor has invested ₵5,000 in the stock market. At the end of the first year the amount has grown to ₵6,250; she has had a 25 percent profit. If at the end of the second year the principal has grown to ₵8,750, the rate of increase is 40 percent for the year. What is the average rate of increase of her investment during the two years?

**Solution**

\[ GM = \sqrt{x_1 \times x_2} = \sqrt{1.25 \times 1.40} = 1.323 \]

The average rate of increase in the value of investment is therefore 1.323 − 1 = 0.323, which if multiplied by 100, gives the rate of increase as 32.3%.

Alternatively, we can also derive a compound interest formula from the above set of data. Now, 1.25 \times 1.40 = 1.75. This can be written as 1.75 = \((1 + 0.323)^2\).

Let \( P_2 = 1.75 \), \( P_0 = 1 \), and \( r = 0.323 \), then the above equation can be written as \( P_2 = (1 + r)^2 \) or \( P_2 = P_0 (1 + r)^2 \), where \( P_2 \) the value of investment at the end of the second year, \( P_0 \) is the initial investment and \( r \) is the rate of increase in the two years.

This, in fact, is the familiar compound interest formula, which can be written in the generalize form as \( P_n = P_0 (1 + r)^n \).

In this case, \( P_0 = 5,000 \) and the rate of increase in investment is \( r = 32.3\% \).

Thus \( P_n = P_0 (1 + r)^n = 5,000 (1 + 0.323)^2 = 8,750 \) is the investment at the end of the 2\(^{nd}\) year.

Note that if the arithmetic mean is used, the resultant figure will be wrong. This is because the average rate for the two years is \( \frac{25 + 40}{2} \) percent per year, which comes to 32.5.

Applying this rate, we get \( P_n = \frac{165}{100} \times 5,000 = 8,250 \)

This is obviously wrong, as the figure should have been ₵8,750.

3. An economy has grown at 5 percent in the first year, 6 percent in the second year, 4.5 percent in the third year, 3 percent in the fourth year and 7.5 percent in the fifth year. What is the average rate of growth of the economy during the five years?

**Solution**

<table>
<thead>
<tr>
<th>Year</th>
<th>Rate</th>
<th>Value x</th>
<th>Log x</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>105</td>
<td>2.02119</td>
</tr>
</tbody>
</table>
2. Consider a stock portfolio that began with a value of ₱1,000 and had annual returns of 13%, 22%, 12%, -5%, and -13%. Use the table below with the value after each of the five years to calculate the geometric mean.

<table>
<thead>
<tr>
<th>Year</th>
<th>Return</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13%</td>
<td>1,130</td>
</tr>
<tr>
<td>2</td>
<td>22%</td>
<td>1,379</td>
</tr>
<tr>
<td>3</td>
<td>12%</td>
<td>1,544</td>
</tr>
<tr>
<td>4</td>
<td>-5%</td>
<td>1,467</td>
</tr>
<tr>
<td>5</td>
<td>-13%</td>
<td>1,276</td>
</tr>
</tbody>
</table>

**Solution**

The question is how to compute average annual rate of return. The answer is to compute the geometric mean of the returns.

Instead of using the percents, each return is represented as a multiplier indicating how much higher the value is after the year.

This multiplier is 1.13 for a 13% return and 0.95 for a 5% loss.

The multipliers are 1.13, 1.22, 1.12, 0.95, and 0.87.

The geometric mean of these multipliers is 1.05.
Therefore, the average annual rate of return is 5%. The portfolio gaining 5% a year would end up with the same value (₵1,276) as shown below.

<table>
<thead>
<tr>
<th>Year</th>
<th>Return</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5%</td>
<td>1,050</td>
</tr>
<tr>
<td>2</td>
<td>5%</td>
<td>1,103</td>
</tr>
<tr>
<td>3</td>
<td>5%</td>
<td>1,158</td>
</tr>
<tr>
<td>4</td>
<td>5%</td>
<td>1,216</td>
</tr>
<tr>
<td>5</td>
<td>5%</td>
<td>1,276</td>
</tr>
</tbody>
</table>

**Advantages of geometric mean**
1. Geometric mean is based on each and every observation in the data set.
2. It is rigidly defined.
3. It is more suitable while averaging ratios and percentages as also in calculating growth rates.
4. As compared to the arithmetic mean, it gives more weight to small values and less weight to large values. As a result of this characteristic of the geometric mean, it is generally less than the arithmetic mean. At times it may be equal to the arithmetic mean.
5. It is capable of algebraic manipulation. If the geometric mean has two or more series is known along with their respective frequencies. Then a combined geometric mean can be calculated by using the logarithms.

**Limitations of geometric mean**
1. As compared to the arithmetic mean, geometric mean is difficult to understand.
2. Both computation of the geometric mean and its interpretation are rather difficult.
3. When there is a negative item in a series or one or more observations have zero value, then the geometric mean cannot be calculated.

In view of the limitations mentioned above, the geometric mean is not frequently used.

**Harmonic Mean (HM)**
The harmonic mean is defined as the reciprocal of the arithmetic mean of the reciprocals of individual observations.

Symbolically, \( HM = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \ldots + \frac{1}{x_n}} = \frac{n}{\sum 1/x_i} \)

The calculation of harmonic mean becomes very tedious when a distribution has a large number of observations. In the case of grouped data, the harmonic mean is calculated by using the following formula:
\[ HM = \text{reciprocal} \sum \left( f_i \times \frac{1}{x_i} \right) = \frac{n}{\sum \left( f_i \times \frac{1}{x_i} \right)} \], Where \( n \) is the total number of observations. Here, each reciprocal of the original figure is weighted by the corresponding frequency \((f)\).

**Example**

1. Suppose we have three observations 4, 8 and 16. Calculate the harmonic mean.

**Solution**

Reciprocals of 4, 8 and 16 are \( \frac{1}{4}, \frac{1}{8} \) and \( \frac{1}{16} \) respectively.

\[ HM = \frac{3}{1/4 + 1/8 + 1/16} = \frac{3}{0.25 + 0.125 + 0.0625} = 6.857 \]

2. Calculate the harmonic means of the data below

<table>
<thead>
<tr>
<th>Class interval</th>
<th>2-4</th>
<th>4-6</th>
<th>6.8</th>
<th>8-10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>20</td>
<td>40</td>
<td>30</td>
<td>10</td>
</tr>
</tbody>
</table>

**Solution**

\[
\begin{array}{|c|c|c|c|c|}
\hline
\text{Class-interval} & \text{Mid-value (x)} & \text{Frequency (f)} & \text{Frequency Reciprocal of x} & f \times 1/x \\
\hline
2-4 & 3 & 20 & 0.3333 & 6.6660 \\
4-6 & 5 & 40 & 0.2000 & 8.0000 \\
6-8 & 7 & 30 & 0.1429 & 4.2870 \\
8-10 & 9 & 10 & 0.1111 & 1.1111 \\
\hline
\text{Total} & & & & 20.0641 \\
\hline
\end{array}
\]

\[ HM = \frac{n}{\sum \left( f_i \times \frac{1}{x_i} \right)} = \frac{100}{20.0641} = 4.984 \]

3. In a small company, two typists are employed. Typist A types one page in ten minutes while typist B takes twenty minutes for the same.

   (i). If both are asked to type 10-page document. What is the average time taken for typing one page?

   (ii). If both are asked to type for one hour. What is the average time taken for typing one page?

**Solution**

Here (i) is on arithmetic mean while (ii) is on harmonic mean.
i. \( \text{Mean} = \frac{10 \times 10 + 20 \times 20}{20 \times 2} = 15 \) minutes

ii. \( \text{Harmonic Mean} = \frac{60}{\frac{60}{10} + \frac{60}{20}} = \frac{40}{3} = 13 \) minutes 20 seconds

4. It takes ship A 10 days to cross the Pacific Ocean; ship B takes 15 days and ship C takes 20 days.
   (i). What is the average number of days taken by a ship to cross the Pacific Ocean?

(ii). What is the average number of days taken by a cargo to cross the Pacific Ocean when the ships are hired for 60 days?

**Solution**

Here again (i) pertains to simple arithmetic mean while (ii) is a harmonic mean.

i. \( \text{Mean} = \frac{10 + 15 + 20}{3} = 15 \) days

ii. \( \text{Harmonic Mean} = \frac{60 \times 3}{\frac{60}{10} + \frac{60}{15} + \frac{60}{20}} = 13.8 \) days

**Advantages of the harmonic mean**

1. The main advantage of the harmonic mean is that it is based on all observations in a distribution and is amenable to further algebraic treatment.

2. When we desire to give greater weight to smaller observations and less weight to the larger observations, then the use of harmonic mean is more suitable.

**Disadvantages of the harmonic mean**

First, it is difficult to understand as well as difficult to compute.

Second, it cannot be calculated if any of the observations is zero or negative.

Third, it is only a summary figure, which may not be an actual observation in the distribution.

Again, it is worth noting that the harmonic mean is always lower than the geometric mean, which is lower than the arithmetic mean. This is because the harmonic mean assigns lesser importance to higher values.

In addition, since the harmonic mean is based on reciprocals, it becomes clear that as reciprocals of higher values are lower than those of lower values, it is a lower average than the arithmetic mean as well as the geometric mean.
**Trimean**
The trimean is a weighted average of the 25th percentile, the 50th percentile, and the 75th percentile. Letting $P_{25}$ be the 25th percentile, $P_{50}$ be the 50th and $P_{75}$ be the 75th percentile, the formula for the trimean is given by:

$$Trimean = \frac{p_{25} + 2p_{50} + p_{75}}{4}$$

The trimean is a robust measure of central tendency; it is a weighted average of the 25th, 50th, and 75th percentiles. Alternatively, it is as: $Trimean = 0.25 \times 25^{th} + 0.5 \times 50^{th} + 0.25 \times 75^{th}$.

**Example**
Given 37, 33, 33, 32, 29, 28, 28, 23, 22, 22, 22, 21, 21, 21, 20, 20, 19, 19, 18, 18, 18, 18, 16, 15, 14, 14, 14, 12, 12, 9, 6. Find the trimean.

**Solution**
The 25th percentile is 15, the 50th is 20 and the 75th percentile is 23.

$$Trimean = \frac{p_{25} + 2p_{50} + p_{75}}{4} = \frac{15 + 2 \times 20 + 23}{4} = 19.5$$

**Trimmed Mean**
To compute a *trimmed mean*, you remove some of the higher and lower scores and compute the mean of the remaining scores.

A mean trimmed 10% is a mean computed with 10% of the scores trimmed off: 5% from the bottom and 5% from the top.

A mean trimmed 50% is computed by trimming the upper 25% of the scores and the lower 25% of the scores and computing the mean of the remaining scores.

The trimmed mean is similar to the median which, in essence, trims the upper 49+% and the lower 49+% of the scores. This means trimmed mean is a robust measure of central tendency generally falling between the mean and the median. And in the computation of the median:

- all observations are ordered.
- Next, the highest and lowest alpha percent of the data are removed, where alpha ranges from 0 to 50.
- Finally, the mean of the remaining observations is taken.
The trimmed mean has advantages over both the mean and media, and therefore a hybrid of the mean and the median, but is analytically more intractable.

**Example**
Find a 20% trimmed mean of the following data: 37, 33, 33, 32, 29, 28, 28, 23, 22, 22, 22, 21, 21, 20, 20, 19, 19, 18, 18, 18, 18, 16, 15, 14, 14, 14, 12, 12, 9, 6.

**Solution**
To compute the mean trimmed 20% for the touchdown pass data above, we first remove the lower 10% of the scores (6, 9, and 12) as well as the upper 10% of the scores (33, 33, and 37).

We then compute the mean of the remaining 25 scores.

This mean is 20.16.

**Activity 5.1**
1. The mean age of 5 persons in a room is 30 years. A 36-year-old person walks in. What is the mean age of the persons in the room now?
2. Out of 100 numbers, 20 were 4s, 40 were 5s, 30 were 6s and the remainder were 7s. Find the arithmetic mean of the numbers.
3. The following set of raw data shows the length, in millimeters, measured to the nearest mm, of each of 40 leaves taken from plants of a certain species. Use the table to find the mean.

<table>
<thead>
<tr>
<th>Length (mm)</th>
<th>Frequency (f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>25-29</td>
<td>2</td>
</tr>
<tr>
<td>30-34</td>
<td>4</td>
</tr>
<tr>
<td>35-39</td>
<td>7</td>
</tr>
<tr>
<td>40-44</td>
<td>10</td>
</tr>
<tr>
<td>45-49</td>
<td>8</td>
</tr>
<tr>
<td>50-54</td>
<td>6</td>
</tr>
<tr>
<td>55-59</td>
<td>3</td>
</tr>
</tbody>
</table>

4. The following set of raw data shows the height, in meters, measured to the nearest m, of each of 20 trees taken from plants of a certain species. Use the table to find the mean.

<table>
<thead>
<tr>
<th>Length (mm)</th>
<th>Frequency (f)</th>
</tr>
</thead>
<tbody>
<tr>
<td>500-600</td>
<td>3</td>
</tr>
<tr>
<td>600-700</td>
<td>6</td>
</tr>
<tr>
<td>700-800</td>
<td>5</td>
</tr>
<tr>
<td>800-900</td>
<td>5</td>
</tr>
<tr>
<td>900-1000</td>
<td>0</td>
</tr>
</tbody>
</table>
5. Calculate the arithmetic mean from the following data

<table>
<thead>
<tr>
<th>Class</th>
<th>10-19</th>
<th>20-29</th>
<th>30-39</th>
<th>40-49</th>
<th>50-59</th>
<th>60-69</th>
<th>70-79</th>
<th>80-89</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>11</td>
<td>12</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

6. A set of numbers is transformed by taking the log base 10 of each number. The mean of the transformed data is 1.65. What is the geometric mean of the untransformed data?

7. Which measure of central tendency is most often used for returns on investment?

**Answers to Activity 5.1**
1. 35
2. 2.20
3. 3

**THE MEDIAN**

**Introduction**
The mean and the median can be calculated to help you find the ‘centre’ of a data set. As the mean is the best estimate for the actual data set, the median is the best measurement when a data set contains several outliers or extreme values, and the mode will tell you the most frequently occurring datum (or data) in the data set.

The three are extremely helpful when you need to analyze your data, but if your data set consists of ranges which lack specific values, the mean may seem impossible to calculate. However, the mean can only be approximated if you add the lower boundary with the upper boundary and divide by two to find the midpoint of each interval, and proceed as outlined in the grouped data.
The ‘centre’ of a data set is also a way of describing location. The two most widely used measures of the ‘centre’ of the data are the mean (average) and the median. We all know that to calculate the mean weight of 50 people, add the 50 weights together and divide by 50. But to find the median weight of the 50 people, we have to order the data and find the number that splits the data into two equal parts.

This makes the median a better measure of the centre when there are extreme values or outliers. This is because it is not affected by the precise numerical values of the outliers.
Various conceptions of the median
The median is the middle number of a set of numbers arranged in numerical order. If the number of values in a set is even, then the median is the sum of the two middle values, divided by 2. The median is not affected by the magnitude of the extreme (smallest or largest) values.

Thus, it is useful because it is not affected by one or two abnormally small or large values, and because it is very simple to calculate. For example, to obtain a relatively accurate average life of a particular type of light bulb, you could measure the median life by installing several bulbs and measuring how much time passed before half of them died. Alternatives would probably involve measuring the life of each bulb.

Median is defined as the value of the middle item (or the mean of the values of the two middle items) when the data are arranged in an ascending or descending order of magnitude. Thus, in an ungrouped frequency distribution if the \( n \) values are arranged in ascending or descending order of magnitude, the median is the middle value if \( n \) is odd. When \( n \) is even, the median is the mean of the two middle values.

The median is the measure used to indicate the centre of a distribution. Ordered elements in the sample, the median is the value (or not the sample belongs) which divides in half, i.e. 50% of the sample elements are less than or equal to the median and 50% are greater than or equal to the median.

The median is a number that separates ordered data into halves so that half the values are the same number or smaller than the median and half the values are the same number or larger than the median. The median may or may not be part of the data at all. When there is an odd number of numbers, the median is simply the middle number, such as the median of 2, 4, and 7 is 4. But when there is an even number of numbers, the median is the mean of the two middle numbers.

The median is a frequently used measure of central tendency. The median is the midpoint of a distribution: the same number of scores is above the median as below it. For the data 37, 33, 33, 32, 29, 28, 28, 23, 22, 22, 22, 21, 21, 20, 20, 19, 19, 18, 18, 18, 18, 16, 15, 14, 14, 14, 12, 12, 9, 6, there are 31 scores. The 16th highest score (which equals 20) is the median because there are 15 scores below the 16th score and 15 scores above the 16th score. The median can also be thought of as the 50th percentile.

Median is defined as the value of the middle item (or the mean of the values of the two middle items) when the data are arranged in an ascending or descending order of magnitude. Thus, in an ungrouped frequency distribution if the \( n \) values are arranged in ascending or descending order of magnitude, the median is the middle value if \( n \) is odd.
When \( n \) is even, the median is the mean of the two middle values. Suppose we have the series: 15, 19,21,7, 10,33,25,18 and 5, we have to first arrange it in either ascending or descending order as: 5,7,10,15,18,19,21,25,33. Because the series consists of odd number of items, we use the formula \( \frac{n+1}{2} \) to obtain 18.

It may be noted that the formula \( \frac{n+1}{2} \) itself is not the formula for the median; it merely indicates the position of the median, namely, the number of items we have to count until we arrive at the item whose value is the median. In the case of the even number of items in the series, we identify the two items whose values have to be averaged to obtain the median.

**Strategies of Computing the Median**

**Median of a discrete data**

The sample median \( \tilde{x} \) of a set of discrete sample data for which there are an odd number of measurements is the middle measurement when the data are arranged in numerical order.

The sample median \( \tilde{x} \) of a set of discrete sample data for which there are an even number of measurements is the mean of the two middle measurements when the data are arranged in numerical order. The median is a value that divides the observations in a data set so that 50% of the data are on its left and the other 50% on its right.

**Example 1**

1. Suppose we have the following series: 15, 19,21,7, 10,33,25,18 and 5. Calculate the median.

   **Solution**

   We have to first arrange it in either ascending or descending order. These figures are arranged in an ascending order as follows: 5,7,10,15,18,19,21,25,33.

   Now as the series consists of odd number of items, to find out the value of the middle item, we use the formula \( \frac{n+1}{2} \). This gives us \( \frac{9+1}{2} = 5 \), that is, the size of the 5th item is the median. This happens to be 18.

2. Suppose the series consists of 5, 7, 10, 15, 18, 19, and 21,23,25,33. Calculate the median.

   **Solution**

   Applying the formula \( \frac{n+1}{2} \), the median is the size of 5.5th item. Here, we have to take the average of the values of 5\(^{th}\) and 6\(^{th}\) item.

   This means an average of 18 and 19, which gives the median as 18.5.
3. Given the variable \( x = \{1, 3, 0, 2.4\} \), calculate the average median of a data set.

**Solution**

i). order the set such as \( x = \{0, 1, 2, 3\} \)

ii). Verify that there is an odd or even number of values in the set; in the above example: 5

iii). If it is odd the median is the value that occupies the central position, and, if even will be the average of the two central positions.

i). Because it is odd, the median is \( \frac{n+1}{2} th = \frac{5+1}{2} = 3rd \), which is 2.

4. Given the data set 2, 4, 7, 12, find the median.

**Solution**

The data has already been ordered

The two middle numbers are 4 and 7

Therefore, the median is \( \frac{4 + 7}{2} = 5.5! \)

**Median of an ungrouped data**

- Rearrange the data in either ascending or descending.
- Find the cumulative frequency of the frequencies.
- Find the cumulative frequency whose sum meets or exceeds the \( \frac{n+1}{2} \) if odd or \( \frac{n}{2} \) if even total number.
- Find the median from the \( \frac{n+1}{2} \) or \( \frac{n}{2} \)

**Example**

1. The following table shows the number of times drivers experience traffic jams in a city. Use the table to find the median.

<table>
<thead>
<tr>
<th>Driving</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Every day</td>
<td>751</td>
</tr>
<tr>
<td>Few days a week</td>
<td>217</td>
</tr>
<tr>
<td>Few days a month</td>
<td>82</td>
</tr>
<tr>
<td>Few times a year</td>
<td>37</td>
</tr>
<tr>
<td>Never</td>
<td>457</td>
</tr>
<tr>
<td>Number</td>
<td>1,544</td>
</tr>
</tbody>
</table>

**Solution**

241
• Find the cumulative frequency of the table.

<table>
<thead>
<tr>
<th>Driving</th>
<th>Frequency</th>
<th>Cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Every day</td>
<td>751</td>
<td>751</td>
</tr>
<tr>
<td>Few days a week</td>
<td>217</td>
<td>968</td>
</tr>
<tr>
<td>Few days a month</td>
<td>82</td>
<td>1,050</td>
</tr>
<tr>
<td>Few times a year</td>
<td>37</td>
<td>1,087</td>
</tr>
<tr>
<td>Never</td>
<td>457</td>
<td>1,544</td>
</tr>
<tr>
<td>Number</td>
<td>1,544</td>
<td>----</td>
</tr>
</tbody>
</table>

• Half of the number is 772.
• The number 772 is found within the cumulative frequency of few days a week (968).
• Therefore, the median is few days a week driving.

**Median of a grouped data**

\[ \frac{n + 1}{2} \text{ if odd or } \frac{n}{2} \text{ if even off the total number} \]

• Find the cumulative frequency whose sum meets or exceeds \( \frac{n + 1}{2} \text{ if odd or } \frac{n}{2} \text{ if even} \) total number.
• Rearrange the data in either ascending or descending.
• Find the class boundaries of the class intervals.
• Find the cumulative frequency of the frequencies.

Example 1. The table shows the wages of workers in a company. Use the table to find the median salary.

<table>
<thead>
<tr>
<th>Monthly wage</th>
<th>Number of workers</th>
</tr>
</thead>
<tbody>
<tr>
<td>800-1,000</td>
<td>18</td>
</tr>
</tbody>
</table>
Solution
In order to calculate median, we have to first provide cumulative frequency as shown below:

<table>
<thead>
<tr>
<th>Monthly wage</th>
<th>Number of workers</th>
<th>Cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>800-1,000</td>
<td>18</td>
<td>18</td>
</tr>
<tr>
<td>1,000-1,200</td>
<td>25</td>
<td>43</td>
</tr>
<tr>
<td>1,200-1,400</td>
<td>30</td>
<td>73</td>
</tr>
<tr>
<td>1,400-1,600</td>
<td>34</td>
<td>107</td>
</tr>
<tr>
<td>1,600-1,800</td>
<td>26</td>
<td>133</td>
</tr>
<tr>
<td>1,800-2,000</td>
<td>10</td>
<td>143</td>
</tr>
<tr>
<td>Total</td>
<td>143</td>
<td></td>
</tr>
</tbody>
</table>

We calculate the \( \frac{n + 1}{2} \) value as \( \frac{143 + 1}{2} = 72 \). So, the median class is 1,200-1,400.

We substitute our values into \( M = L_1 + \frac{L_2 - L_1}{f} (m - c) \) as

\[
M = 1,200 + \frac{1,400 - 1,200}{30} (72 - 43) = 1,393.30
\]

2. Calculate the most suitable average for the following data:

<table>
<thead>
<tr>
<th>Size</th>
<th>Below 50</th>
<th>50-100</th>
<th>100-150</th>
<th>150-200</th>
<th>200 and above</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>15</td>
<td>20</td>
<td>36</td>
<td>40</td>
<td>10</td>
</tr>
</tbody>
</table>

Solution
Since the data have two open-end classes—one in the beginning (below 50) and the other at the end (200 and above), median should be the right choice as a measure of central tendency.

So, we find the cumulative frequency table as below:

<table>
<thead>
<tr>
<th>Size</th>
<th>Frequency</th>
<th>Cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>Below 50</td>
<td>15</td>
<td>15</td>
</tr>
<tr>
<td>50-100</td>
<td>20</td>
<td>35</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>----------</td>
<td>------</td>
<td>------</td>
</tr>
<tr>
<td>100-150</td>
<td>36</td>
<td>71</td>
</tr>
<tr>
<td>150-200</td>
<td>40</td>
<td>111</td>
</tr>
<tr>
<td>200 and above</td>
<td>10</td>
<td>121</td>
</tr>
<tr>
<td>Total</td>
<td>121</td>
<td></td>
</tr>
</tbody>
</table>

Calculate the median class of \(\frac{n+1}{2} = \frac{121+1}{2} = 61\), which lies between 100-150.

Calculate the value of \(M = L_1 + \frac{L_2 - L_1}{f} (m - c) = 100 + \frac{150 - 100}{36} (61 - 35) = 136.11\)

**Questions**

1. The data in numerical order are 1.39; 1.76; 1.90; 2.12; 2.53; 2.71; 3.00; 3.33; 3.71; 4.00. Calculate the median.

**Solution**
The number of observations is ten, which is even, so there are two middle measurements, the fifth and sixth, which are 2.53 and 2.71. Therefore the median of these data is \(M = \frac{2.53 + 2.71}{2} = 2.62\).

2. Suppose that in a small town of 50 people, one person earns ₦5,000,000 per year and the other 49 each earn ₦30,000. Which is the better measure of the ‘centre’: the mean or the median?

**Solution**
\[\bar{x} = \frac{5,000,000 + 49(30,000)}{50} = 129,400\] and \(M = 30,000\)

There are 49 people who earn ₦30,000 and one person who earns ₦5,000,000. The median is a better measure of the ‘centre’ than the mean because 49 of the values are 30,000 and one is 5,000,000. The 5,000,000 is an outlier.

The 30,000 gives us a better sense of the middle of the data.

3. The playbill for the Alley Theatre in Houston wants to appeal to advertisers. They reported the mean household income and the median age of theatre goers. What might have guided their choice of the mean or median?

**Solution**
It is likely that they wanted to emphasize that theatre goers had high income but de-emphasize how old they are. The distributions of income and age of theatre goers probably have positive
Therefore the mean is probably higher than the median, which results in higher income and lower age than if the median household income and mean age had been presented!

**Characteristics of the Median**
1. Unlike the arithmetic mean, the median can be computed from open-ended distributions. This is because it is located in the median class-interval, which would not be an open-ended class.
2. The median can also be determined graphically whereas the arithmetic mean cannot be ascertained in this manner.
3. As it is not influenced by the extreme values, it is preferred in case of a distribution having extreme values.
4. In case of the qualitative data where the items are not counted or measured but are scored or ranked, it is the most appropriate measure of central tendency.

**Advantages of the median**
- **Simplicity:** It is very simple measure of the central tendency of the series. In the case of simple statistical series, just a glance at the data is enough to locate the median value.
- **Free from the effect of extreme values:** Unlike arithmetic mean, median value is not destroyed by the extreme values of the series.
- **Certainty:** Certainty is another merits is the median. Median values are always a certain specific value in the series.
- **Real value:** Median value is real value and is a better representative value of the series compared to arithmetic mean average, the value of which may not exist in the series at all.
- **Graphic presentation:** Besides algebraic approach, the median value can be estimated also through the graphic presentation of data.
- **Possible even when data is incomplete:** Median can be estimated even in the case of certain incomplete series. It is enough if one knows the number of items and the middle item of the series.
- **Open end intervals:** As taking any value of the intervals, value of the median remains the same.
- **Other statistical devices:** The median is also used for other statistical devices such as mean deviation and skewness.
- It can be used for the Quantities. It is possible to arrange in any order and to locate the middle value. For such cases it is the best measure.
- It can easily located by inspection
- It is appropriate for a qualitative data

**Disadvantages of the median**
• **Lack of representative character**: Median fails to be a representative measure in case of such series the different values of which are wide apart from each other.

• **Limited representative character**: also, median is of limited representative character as it is not based on all the items in the series.

• **Unrealistic**: When the median is located somewhere between the two middle values, it remains only an approximate measure, not a precise value.

• **Lack of algebraic treatment**: Arithmetic mean is capable of further algebraic treatment, but median is not. For example, multiplying the median with the number of items in the series will not give us the sum total of the values of the series.

• **Unequal class intervals**: It cannot be used if the class intervals are unequal.

• **Fluctuation of data**: It is more affected by fluctuation of the data than the mean.

• **Extreme items**: Even if the value of extreme items is too large, it does not affect too much, but due to this reason, sometimes median does not remain the representative of the series.

• **Continuous series**: In a continuous series it has to be interpolated. We can find its true-value only if the frequencies are uniformly spread over the whole class interval in which median lies.

• Even series: If the number of series is even, we can only make its estimate; as the mean of two middle terms is taken as Median

**Exercise 5.2**

1. Design three data sets on your own that have 5 numbers in each set so that:
   (a). the same mean but different standard deviations.
   (b). the same mean but different medians.
   (c). the same median but different means.

2. Compare the mean, median, trimean in terms of their sensitivity to extreme scores.

3. If the mean time to respond to a stimulus is much higher than the median time to respond, what can you say about the shape of the distribution of response times?

4. Find the mean and the median for the LDL cholesterol level in a sample of ten heart patients of 132, 139, 162, 147, 133, 160, 145, 150, 148, 153.

   **Answers**
   4. \( \bar{x} = 146.9 \) \quad M = 147.5

5. Find the mean and the median, for the LDL cholesterol level in a sample of ten heart patients on a special diet of 127, 113, 152, 131, 138, 148, 110, 135, 152, 158
6. A man tosses a coin repeatedly until it lands heads and records the number of tosses required. (For example, if it lands heads on the first toss he records a 1; if it lands tails on the first two tosses and heads on the third he records a 3.) The data are shown.

<table>
<thead>
<tr>
<th>X</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>384</td>
<td>208</td>
<td>98</td>
<td>56</td>
<td>28</td>
<td>12</td>
<td>8</td>
<td>2</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

a. Find the mean of the data.
b. Find the median of the data.

**Answers**
6. $\bar{x} = 2.05$, $Me = 2$, $Mo = 1$

8. In a sample of 60 households, one house is worth ₵2,500,000. Half of the rest are worth ₵280,000, and all the others are worth ₵315,000. Which is the better measure of the ‘centre’: the mean or the median? ([Stats1pp1107](Stats1pp1107)).

**Summary**
How do the various measures of central tendency compare with each other? For symmetric distributions, the mean, median, trimean, and trimmed mean are equal, as is the mode except in bimodal distributions. Differences among the measures occur with skewed distributions.

In the distribution, 37, 33, 33, 32, 29, 28, 28, 23, 22, 22, 22, 21, 21, 21, 20, 20, 19, 19, 18, 18, 18, 18, 16, 15, 14, 14, 14, 12, 12, 9, 6, we can notice they do not differ greatly, with the exception that the mode is considerably lower than the other measures.

When distributions have a positive skew, the mean is typically higher than the median, although it may not be in bimodal distributions. For these data, the mean of 91.58 is higher than the median of 90.

Typically, the trimean and trimmed mean will fall between the median and the mean, although in this case, the trimmed mean is slightly lower than the median. The geometric mean is lower than all measures except the mode ([Stats5](Stats5)).
There is no need to summarize a distribution with a single number. When the various measures differ, our opinion is that you should report the mean, median, and either the trimean or the mean trimmed 50%. Sometimes it is worth reporting the mode as well.

In the media, the median is usually reported to summarize the centre of skewed distributions. You will hear about median salaries and median prices of houses sold, and so on. This is better than reporting only the mean, but it would be informative to hear more statistics (Stats5).

**THE MODE**

**Introduction**

Another measure of the centre is the mode. The mode is the most frequent value. There can be more than one mode in a data set as long as those values have the same frequency and that frequency is the highest.

A data set with one mode is called unimodal, with two modes is called bimodal and with three modes is called trimodal. Any other number of modes beyond three is called multimodal.

The mode is also a measure of central location since most real-life data sets have more observations near the centre of the data range and fewer observations on the lower and upper ends. The value with the highest frequency is often in the middle of the data range.

The mode can be calculated for qualitative data as well as for quantitative data. For instance, in a qualitative data such as red, red, red, green, green, yellow, purple, black, blue, the mode is red.
And in a quantitative data such as 5, 3, 6, 5, 4, 5, 2, 8, 6, 5, 4, 8, 3, 4, 5, 4, 8, 2, 5, and 4, the mode is 5.

**Conceptions of the mode**
The mode is the value at the point around which the items are most heavily concentrated.

The mode is the most frequent value in a distribution: the mode of 3, 4, 4, 5, 5, 5, 8 is 5. Note that the mode may be very different from the mean and the median.

The sample mode of a set of sample data is the most frequently occurring value. On a relative frequency histogram, the highest point of the histogram corresponds to the mode of the data set.

**Strategies of Computing the Arithmetic Mode**

**Mode for discrete ungrouped data**
Mode is the value that occurs most frequently.

**Example 1**
1. Statistics exam scores for 20 students are as follows:
50; 53; 59; 59; 63; 63; 72; 72; 72; 72; 72; 76; 78; 81; 83; 84; 84; 84; 90; 93. Find the mode.

**Solution**
The most frequent score is 72, which occurs five times.
Therefore, the mode is 72.
2. The number of books checked out from the library from 25 students are as follows:
0; 0; 0; 1; 2; 3; 3; 4; 4; 5; 5; 7; 7; 7; 7; 8; 8; 8; 9; 10; 10; 11; 11; 12; 12. Find the mode.

**Solution**
The mode is 7

3. Five real estate exam scores are 430, 430, 480, 480, and 495. Find the mode.

**Solution**
The data set is bimodal because the scores 430 and 480 each occur twice.
Therefore, the modes are 430 and 480.

4. Five credit scores are 680, 680, 700, 720, and 720. Find the mode.

**Solution**
The data set is bimodal because the scores 680 and 720 each occur twice.
Therefore, the modes are 680 and 720.

5. Create a sample data set of size \( n = 4 \) for which the mean, the median, and the mode are all identical.
Solution
\{0, 1, 1, 2\}

Mode for grouped data

With continuous data, such as response time measured to many decimals, the frequency of each value is one since no two scores will be exactly the same. Therefore, the mode of continuous data is normally computed from a grouped frequency distribution.

Statistical method

In order to calculate the mode of grouped data:

- Find the modal class---the class interval that has the largest frequency.
- Find the lower class boundary of the modal class (lb).
- Find the difference of frequency between the modal class to its upper class (a).
- Find the difference of frequency between the modal class to its lower class (b).
- Add lb to the to products \( \frac{a}{a + b} \) by c.
- Therefore, \( Mo = Lo + \frac{a}{a + b} \times C \) where Lo is the lower limit of the modal class
  b is frequency of the modal class - frequency of previous class to the modal class
  a is frequency of the modal class - frequency of posterior class to the modal class
  C is amplitude of the modal class

Alternatively, the mode can be determined by:

\[
Mo = L_1 + \frac{f_1 - f_0}{(f_1 - f_0) + (f_1 - f_2)} \times C
\]

Where, \( L_1 \) is the lower value of the class in which the mode lies
  \( f_i \) is the frequency of the class in which the mode lies
  \( f_0 \) is the frequency of the class preceding the modal class
  \( f_2 \) is the frequency of the class succeeding the modal class
  \( C \) is the class-interval of the modal class

Example 2

Find the mode from the table below:
We can see from Column (2) of the table that the maximum frequency of 12 lies in the class-interval of 60-70. This suggests that the mode lies in this class interval.

Applying the second formula given earlier, we get:

\[ Mo = L_a + \frac{f_1 - f_0}{(f_1 - f_0) + (f_1 - f_2)} \times C = 60 + \frac{12 - 8}{(12 - 8) + (12 - 9)} \times 10 \]

\[ Mo = 60 + \frac{12 - 8}{(12 - 8) + (12 - 9)} \times 10 = 60 + \frac{4}{4 + 3} \times 10 = 65.7 \]

**Algebraic method**

If we have a bi-modal or multimodal series, which shows that some classes have occurred the same number of times, we can determine mode indirectly by applying the following formula:

\[ Mo = 3\text{median} - 2\text{mean} = 3M - 2\bar{x} \]

**Example 3**

Find the mode from the table below:

<table>
<thead>
<tr>
<th>Size</th>
<th>10-20</th>
<th>20-30</th>
<th>30-40</th>
<th>40-50</th>
<th>50-60</th>
<th>60-70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>10</td>
<td>18</td>
<td>25</td>
<td>26</td>
<td>17</td>
<td>4</td>
</tr>
</tbody>
</table>

**Solution**

<table>
<thead>
<tr>
<th>Size</th>
<th>Frequency</th>
<th>cf</th>
<th>x</th>
<th>d</th>
<th>u=d/10</th>
<th>fu</th>
</tr>
</thead>
<tbody>
<tr>
<td>10-20</td>
<td>10</td>
<td>10</td>
<td>15</td>
<td>-20</td>
<td>-2</td>
<td>-20</td>
</tr>
<tr>
<td>20-30</td>
<td>18</td>
<td>28</td>
<td>25</td>
<td>-10</td>
<td>-1</td>
<td>-18</td>
</tr>
<tr>
<td>30-40</td>
<td>25</td>
<td>53</td>
<td>35</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>40-50</td>
<td>26</td>
<td>79</td>
<td>45</td>
<td>10</td>
<td>1</td>
<td>26</td>
</tr>
<tr>
<td>50-60</td>
<td>17</td>
<td>96</td>
<td>55</td>
<td>20</td>
<td>2</td>
<td>34</td>
</tr>
<tr>
<td>60-70</td>
<td>4</td>
<td>100</td>
<td>65</td>
<td>30</td>
<td>3</td>
<td>12</td>
</tr>
</tbody>
</table>
### Relationship between the Mean, Median, and Mode

#### Symmetric data (Stats1pp111)

A distribution is symmetrical if a vertical line can be drawn at some point in the histogram such that the shape to the left and the right of the vertical line are mirror images of each other.

The mean, the median, and the mode are each seven for these data. In a perfectly symmetrical distribution, the mean and the median are the same.

This example has one mode (unimodal), and the mode is the same as the mean and median. In a symmetrical distribution that has two modes (bimodal) or more modes (multimodal), the modes are normally different from the mean and median.

#### Left skewed data

The right-hand side seems ‘chopped off’ compared to the left side. A distribution of this type is called skewed to the left because it is pulled out to the left.

---

| Total | 100 | 34 |

\[
\text{Median} = 30 + \frac{40 - 30}{25} \times (50.5 - 28) = 39
\]

\[
\text{Mean} = 35 + \frac{34}{100} \times 10 = 38.4
\]

\[
\text{Mode} = 3(39) - 2(38.4) = 40.2
\]
We can see that the mean is 6.3, the median is 6.5, and the mode is 7. Notice that the mean is less than the median, and they are both less than the mode. The mean and the median both reflect the skewing, but the mean reflects it more so.

Therefore, when a distribution is skewed to the left, then \( mode > median > mean \). This is because the mean is pulled down below the median by extremely low values.

**Right skew data**
The left-hand side seems ‘chopped off’ compared to the right side. A distribution of this type is called skewed to the right because it is pulled out to the right.
In this figure, the mean is 7.7, the median is 7.5, and the mode is 7. Of the three statistics, the mean is the largest, while the mode is the smallest. Again, the mean reflects the skewing the most.

If the distribution of data is skewed to the right, the mode is often less than the median, which is less than the mean. That is, \( \text{mean} > \text{median} > \text{mode} \). In such a case, the mean is pulled up by the extreme high values.

When \( \text{mean} > \text{median} \), it is skewed to the right; when \( \text{median} > \text{mean} \), it is skewed to the left. It may be noted that the median is always in the middle between mean and mode.

**The Best Measure of Central Tendency**
The question commonly asked is: which of the three measures of central tendency is the best? There is no simple answer to this question. This is because the three measures are based upon different concepts:

- The arithmetic mean is the sum of the values divided by the total number of observations in the series.
- The median is the value of the middle observation that divides the series into two equal parts.
- Mode is the value around which the observations tend to concentrate.

As such, the use of a particular measure will largely depend on the purpose of the study and the nature of the data;

- When we are interested in knowing the consumers preferences for different brands of television sets or different kinds of advertising, the choice should go in favour of mode. The use of mean and median would not be proper.
- Median can sometimes be used in the case of qualitative data when such data can be arranged in an ascending or descending order.
- Mean is most appropriate in the case of quantitative data when such data can be arranged in an interval or a ratio.

**Example 4**
1. Find the mean, the median, and the mode for the number of vehicles owned in a survey of 52 households in the following table:

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>2</td>
<td>12</td>
<td>15</td>
<td>11</td>
<td>6</td>
<td>3</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

**Solution**
Mean=2.6, median=2, mode=2
2. Begin with the following set of Data Set I: 5, −2, 6, 14, −3, 0, 1, 4, 3, 2, 5.
   a. Compute the mean, median, and mode.
   b. Form a new data set, Data Set II, by adding 3 to each number in Data Set I. Calculate the mean, median, and mode of Data Set II.
   c. Form a new data set, Data Set III, by subtracting 6 from each number in Data Set I. Calculate the mean, median, and mode of Data Set III.
   d. Comparing the answers to parts (a), (b), and (c), can you guess the pattern? State the general principle that you expect to be true.

**Answers**

2. a. \( \bar{x} = 3.18 \), \( Me = 3 \), \( Mo = 5 \)
   b. \( \bar{x} = 6.18 \), \( Me = 6 \), \( Mo = 8 \)
   c. \( \bar{x} = -2.18 \), \( Me = -3 \), \( Mo = -1 \)

3. Suppose we invite applications for a certain vacancy in our company. A large number of candidates apply for that post. What measure should we use to determine the age or age group with the largest concentration of applicants?

**Solution**

Here, obviously the mode will be the most appropriate choice. The arithmetic mean may not be appropriate as it may be influenced by some extreme values. However, the mean happens to be the most commonly used measure of central tendency as will be evident from other data.

**Exercise**

1. Calculate the mean, median and mode from the following data:
   a. 
<table>
<thead>
<tr>
<th>Class</th>
<th>10-19</th>
<th>20-29</th>
<th>30-39</th>
<th>40-49</th>
<th>50-59</th>
<th>60-69</th>
<th>70-79</th>
<th>80-89</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>2</td>
<td>4</td>
<td>9</td>
<td>11</td>
<td>12</td>
<td>6</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>
   b. 
   | Height (inc) | 62-63 | 63-64 | 64-65 | 65-66 | 66-67 | 67-68 | 68-69 |

255
2. What are the requirements of a good average? Compare the mean, the median and the mode in the light of these requirements? Why averages are called measures of central tendency?

Summary
We have learned that the mean, the median, and the mode each answer the question “Where is the centre of the data set?” The nature of the data set determines which one gives the best answer. It is left with the learner to determine the best method to calculate the central value.

UNIT 5 SECTION 5 TEACHING LOCATIONS
Introduction
Dear Learner,
We have learned that the mean, mode and median describe the centre of a distribution. The range and the standard deviation describe the variability of the distribution. One type of measure of position tells us the point where a certain percentage of the data fall above or fall below that point.

The median is an example that specifies a location such that half the data fall below it and half fall above it. The range uses two other measures of position, the maximum value and the minimum value. The z-score tells us how far an observation falls from a particular point, such as the number of standard deviations an observation falls from the mean.

Another more common use of the term quantiles is a general term for partitioning ranked data into equal parts. For example, quartiles partition the data into 4 equal parts. Percentiles partition the data into 100 equal parts. Thus, the k-th q-tile is the value in the data for which k/q of the values are below the given value. This naturally leads to some rounding issues which lead to a large variety of small differences in the definition of quantiles.

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The common measures of location are quartiles and percentiles. Quartiles are special percentiles. The first quartile, $Q_1$, is the same as the 25th percentile, and the third quartile, $Q_3$, is the same as the 75th percentile. The median, $M$, is called both the second quartile and the 50th percentile.
To calculate quartiles and percentiles, the data must be ordered from smallest to largest. Quartiles divide ordered data into quarters. Percentiles divide ordered data into hundredths. To score in the 90th percentile of an exam does not mean, necessarily, that you received 90% on a test. It means that 90% of test scores are the same or less than your score and 10% of the test scores are the same or greater than your test score.

**Locations of percentiles**
The values that divide a rank-ordered set of data into 100 equal parts are called percentiles. Percentiles are used to compare and interpret data. For example, an observation at the 50th percentile would be greater than 50 percent of the other observations in the set.

Percentiles are useful for comparing values. For this reason, universities and colleges use percentiles extensively. One instance in which colleges and universities use percentiles is when exam results are used to determine a minimum testing score that will be used as an acceptance factor. For example, suppose UEW accepts exam scores at or above the 75th percentile for postgraduate studies. That translates into a score of at least 3.50.

Percentiles are mostly used with very large populations. Therefore, if you were to say that 90% of the test scores are less (and not the same or less) than your score, it would be acceptable because removing one particular data value is not statistically significant.

The Pth percentile cuts the data set in two so that approximately P% of the data lie below it and (100 − P)% of the data lie above it. In particular, the three percentiles that cut the data into fourths are called the quartiles.

There is no universally accepted definition of a percentile. Using the 65th percentile as an example:
1. The 65th percentile can be defined as the lowest score that is greater than 65% of the scores.
2. The 65th percentile can also be defined as the smallest score that is greater than or equal to 65% of the scores.
3. The 65th percentile can also be defined as a weighted average of the percentiles computed according to the first two definitions.

Unfortunately, the first two definitions can lead to dramatically different results, especially when there is relatively little data.

Moreover, neither of these definitions is explicit about how to handle rounding. For instance, what rank is required to be higher than 65% of the scores when the total number of scores is 50?
This is tricky because 65% of 50 is 32.5. How do we find the lowest number that is higher than 32.5% of the scores? The third definition handles rounding more gracefully than the other two and has the advantage that it allows the median to be defined conveniently as the 50th percentile.

In all cases, the position of the percentile is \( i = \left( \frac{k}{100} \right)(n+1) \), where:

- \( k \) = the \( k \)th percentile. It may or may not be part of the data.
- \( i \) = the index (ranking or position of a data value)
- \( n \) = the total number of data

The procedures are:

- Order the data from smallest to largest.
- Calculate \( R_i = \left( \frac{k}{100} \right)(n+1) \)
- If \( i \) is an integer, then the \( k \)th percentile is the data value in the \( i \)th position in the ordered set of data.
- If \( i \) is not an integer, then round \( i \) up and round \( i \) down to the nearest integers. Average the two data values in these two positions in the ordered data set.

**Strategies of Calculating Percentiles**

**Discrete data of percentile**

We can use the above formula for the calculation of quartiles as well. The only difference will be in the value of \( m \). Let us calculate both Q1 and Q3 in respect of the table given below.

<table>
<thead>
<tr>
<th>Wages</th>
<th>800-1000</th>
<th>1000-1200</th>
<th>1200-1400</th>
<th>1400-1600</th>
<th>1600-1800</th>
<th>1800-2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Workers</td>
<td>18</td>
<td>25</td>
<td>30</td>
<td>34</td>
<td>26</td>
<td>10</td>
</tr>
</tbody>
</table>

**Solution**

\[
Q_1 = l_1 + \frac{l_2 - l_1}{f}(m - c)
\]

In the case of Q1, \( m = \frac{1(n+1)}{4} = \frac{143+1}{4} = 36 \)

\[
Q_1 = l_1 + \frac{l_2 - l_1}{f}(m - c) = 1000 + \frac{1200-1000}{25}(36-18) = 1144
\]

In the case of Q3, \( m \) will be 3: \( m = \frac{3(n+1)}{4} = \frac{3(144)}{4} = 108 \)

\[
Q_3 = l_1 + \frac{l_2 - l_1}{f}(m - c) = 1600 + \frac{1800-1600}{25}(108-107) = 1607.70
\]
Example
1. Find the 25th percentile for the 8 numbers in the Table below.

<table>
<thead>
<tr>
<th>Number</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>11</th>
<th>13</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rank</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td>8</td>
</tr>
</tbody>
</table>

Solution
The first step is to compute the rank (R) of the 25th percentile. This is done using the formula:

\[ R = \frac{P}{100} \times (n + 1), \]

where P is the desired percentile (25 in this case) and n is the number of numbers (8 in this case).

Therefore, \[ R = \frac{25}{100} \times (8 + 1) = 2.25. \]

If R is an integer, the Pth percentile is the number with rank R.

When R is not an integer, we compute the Pth percentile by interpolation as follows:

1. Define IR as the integer portion of R (the number to the left of the decimal point i.e. IR=2).
2. Define FR as the fractional portion of R (the number to the right of the decimal point i.e. FR=0.25).
3. Find the scores with Rank IR and with Rank IR + 1. This means the score with Rank 2 and the score with Rank 3 are 5 and 7 respectively.
4. Interpolate by multiplying the difference between the scores by FR and add the result to the lower score. That is, \[ (0.25) \times (7 - 5) + 5 = 5.5. \]

Therefore, the 25th percentile is 5.5.

- If we had used the first definition (the smallest score greater than 25% of the scores), the 25th percentile would have been 7.
- If we had used the smallest score greater than or equal to 25% of the scores, the 25th percentile would have been 5.
- But because we used the weighted average of the percentiles computed according to the first two definitions, the 25th percentile is 5.5.

2. The table below represents the cores of 20 pupils in a test. Use the table to find the 25th and 85th percentiles.

| Score | 4  | 4  | 5  | 5  | 5  | 5  | 6  | 6  | 6  | 7  | 7  | 8  | 8  | 8  | 9  | 9  | 9  | 10 | 10 | 10 | 10 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Rank  | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |

Solution
a). For the 25th:
\[
R = \frac{p}{100} \times (n + 1) = \frac{25}{100} \times (25 + 1) = 5.25
\]

IR = 5 and FR = 0.25.

Since the score with a rank of IR (which is 5) and the score with a rank of IR + 1 (which is 6) are both equal to 5, the 25th percentile is 5.

In terms of the formula: \( R_{25} = (0.25) \times (5 - 5) + 5 = 5 \).

b). For the 85th percentile:

\[
R = \frac{p}{100} \times (n + 1) = \frac{85}{100} \times (20 + 1) = 17.85
\]

IR = 17 and FR = 0.85

Caution: FR does not generally equal the percentile to be computed as it does here.

The score with a rank of 17 is 9 and the score with a rank of 18 is 10. Therefore, the 85th percentile is: \( R_{85} = (0.85) \times (10 - 9) + 9 = 9.85 \)

3. Find the 50th percentile of the numbers 2, 3, 5, 9.

Solution

\[
R = \frac{p}{100} \times (n + 1) = \frac{50}{100} \times (4 + 1) = 2.5
\]

IR = 2 and FR = 0.5.

The score with a rank of IR is 3 and the score with a rank of IR + 1 is 5.

Therefore, the 50th percentile is: \( R_{50} = (0.5) \times (5 - 3) + 3 = 4 \)

4. Find the 50th percentile of the numbers 2, 3, 5, 9, 11.

Solution

\[
R = \frac{p}{100} \times (n + 1) = \frac{50}{100} \times (5 + 1) = 3.0
\]

IR = 3 and FR = 0.

Whenever FR = 0, we simply find the number with rank IR. In this case, the third number is equal to 5.

So, the 50th percentile is 5 or \( R_{50} = (0.00) \times (9 - 5) + 5 = 5 \)

4. Listed are 29 ages for Academy Award winning best actors in order from smallest to largest. 18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77
a. Find the 70th percentile.
b. Find the 83rd percentile.

Solution
a. \( k = 70 \)
i is the index
\( n = 29 \)
\[ i = \left( \frac{k}{100} \right) (n + 1) = \frac{70}{100} (29 + 1) = 21 \]

b. \( k = 83 \)
i is the index
\( n = 29 \)
\[ i = \left( \frac{k}{100} \right) (n + 1) = \frac{83}{100} (29 + 1) = 24.9 \text{, which is NOT an integer. Round it down to 24 and up to 25.} \]

The age in the 24th position is 71 and the age in the 25th position is 72. Average of 71 and 72 is 71.5 years.

A formula for finding the percentile of a value in a data set
- Order the data from smallest to largest.
- \( x \) = the number of data values counting from the bottom of the data list up to but not including the data value for which you want to find the percentile.
- \( y \) = the number of data values equal to the data value for which you want to find the percentile.
- \( n \) = the total number of data.
- Calculate \( \left( \frac{x + 0.5y}{n} \right) \times 100 \).

Then round to the nearest integer.

Example 2
1. Listed are 29 ages for Academy Award winning best actors in order from smallest to largest.
18; 21; 22; 25; 26; 27; 29; 30; 31; 33; 36; 37; 41; 42; 47; 52; 55; 57; 58; 62; 64; 67; 69; 71; 72; 73; 74; 76; 77
   a. Find the percentile for 58.
   b. Find the percentile for 25.
Solution
a. Counting from the bottom of the list, there are 18 data values less than 58. There is one value of 58.

\[ x = 18 \text{ and } y = 1 \left(\frac{x + 0.5y}{n}\right) (100) = \frac{18 + 0.5(1)}{29} (100) = 63.80 \]

58 is the 64th percentile.

b. Counting from the bottom of the list, there are three data values less than 25. There is one value of 25.

\[ x = 3 \text{ and } y = 1 \left(\frac{x + 0.5y}{n}\right) (100) = \frac{3 + 0.5(1)}{29} (100) = 12.07 \]

Twenty-five is the 12th percentile.

In the same manner, we can calculate deciles (where the series is divided into 10 parts) and percentiles (where the series is divided into 100 parts) (Stats10pp32).

Locations of Deciles or Decision

!---!---!---!---!---!---!---!---!---!---!

D1 D2 D3 D4 D5 D6 D7 D8 D9

These are values that divide the ordered data set (list) within ten (10) equal parts.

First Decile (D1) - set value so the data series that 10% of the observations are smaller than him and 90% are greater.

Second Decile (D2) - set value so the data series that 20% of the observations are smaller than him and 80% are greater.

Ninth Decile (D9) - set value so the data series that 90% of the observations are smaller than him and 10% are greater.

Steps taken to calculate the First decile:

- Determined the n (adding column fi);
- Calculate the value of (n / 10) (whether n is even or odd!);
- Build the cumulative frequency column;

Compare the value of (n / 10) with the cumulative values, starting from the first class (the top!) and asking the question: “This cumulative is greater than or equal to (n / 10)?” If the answer is NO, the cumulative goes to the next class. When the answer is YES, we will stop and try the corresponding class!
Finally, we will apply the formula of Q3, extracting the data Q1 of the class, and use the formula.

**Example**

For the set below, determine the value of the first decile.

<table>
<thead>
<tr>
<th>Xi</th>
<th>0-10</th>
<th>10-20</th>
<th>20-30</th>
<th>30-40</th>
<th>40-50</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>fi</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>24</td>
</tr>
</tbody>
</table>

**Solution**

Step 1) we will find and calculate \((n / 10)\): Hence, we find that \(n = 24\) and therefore \((n / 10) = 2.4\)

Steps 2) build the cumulative frequency column.

Step 3) we compared the values with the value of \((n / 10)\):

<table>
<thead>
<tr>
<th>Xi</th>
<th>0-10</th>
<th>10-20</th>
<th>20-30</th>
<th>30-40</th>
<th>40-50</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>fi</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>24</td>
</tr>
<tr>
<td>cf</td>
<td>2</td>
<td>7</td>
<td>15</td>
<td>21</td>
<td>24</td>
<td></td>
</tr>
<tr>
<td>Clues</td>
<td>&gt;2.4, No</td>
<td>&gt;2.4, Yes</td>
<td>&gt;2.4, Yes</td>
<td>&gt;18, Yes</td>
<td>&gt;18, Yes</td>
<td></td>
</tr>
</tbody>
</table>

We think, therefore, that the corresponding class (10-20) will be our Class of First Decile!

Step 4) we apply the formula of the First Decile:

To the E: \(D_1 = 10.8\)

**Locations of quartiles or Separatrices**
A safer procedure is to obtain a type of range that uses observations that are not the largest and smallest values. The *interquartile range* (IQR) is one that is commonly used. The interquartile range is defined as \( \text{IQR} = Q_3 - Q_1 \).

Three quartiles divide the distribution into four equal parts, with 25% of the distribution in each part. We have already introduced one of the quartiles, Q2, which is the median. The quartile Q1 divides the lower half of the distribution into halves; Q3 divides the upper half of the distribution into halves. Quartiles are computed by first ordering the data, and the location of Q1 is \( .25(n+1) \), Q2 is \( .50(n+1) \), and Q3 is \( .75(n+1) \).

The interquartile range is available in many statistical programs. The Q1 and Q3 quartiles are not easy measures to compute by hand, as they often require interpolation. (Interpolation is a method of estimating an unknown value by using its position among a series of known values. Since the quartiles are not sensitive to the numerical values of extreme observations, they are considered measures of location resistant to the effect of outliers.

Note that the numerical value of the difference between the median and Q1 does not have to equal the difference between Q3 and the median. If the distribution is skewed to the right, then Q3 minus the median usually is larger than the median minus Q1. But the proportion of observations is the same.

For small samples, fourths are simpler measures to compute. If \( n \) is even, we simply compute the median of the lower and upper halves of the ordered observations and call them the lower and upper fourths, respectively. If \( n \) is odd, we consider the middle measurement, or median, to be part
of the lower half of measurements and compute the median of these measurements, Q1. Then assign the median to the upper half of the measurements and compute the median of the upper half, Q3.

Quartiles are numbers that separate the data into quarters. Quartiles may or may not be part of the data. To find the quartiles, first find the median or second quartile. The first quartile, Q1, is the middle value of the lower half of the data, and the third quartile, Q3, is the middle value, or median, of the upper half of the data. To get the idea, consider the same data set:

1; 1; 2; 2; 4; 6; 6.8; 7.2; 8; 8.3; 9; 10; 10; 11.5.

The number two, which is part of the data, is the first quartile. One-fourth of the entire sets of values are the same as or less than two and three-fourths of the values are more than two. The upper half of the data is 7.2, 8, 8.3, 9, 10, 10, 11.5. The middle value of the upper half is nine.

Quartiles divide data into quarters. The first quartile (Q1) is the 25th percentile, the second quartile (Q2 or median) is 50th percentile, and the third quartile (Q3) is the 75th percentile. The interquartile range, or IQR, is the range of the middle 50 percent of the data values. The IQR is found by subtracting Q1 from Q3, and can help determine outliers by using the following two expressions.

• $Q3 + IQR(1.5)$
• $Q1 - IQR(1.5)$

Expression for finding the percentile of a data value:

$$i = \left( \frac{k}{100} \right) (n + 1),$$

where $i$ is the ranking or position of a data value, $k$ is the kth percentile, $n$ is the total number of data.

To understand quartile and decile, we should first know that the median belongs to a general class of statistical descriptions called fractiles. A fractile is a value below that lays a given fraction of a set of data. In the case of the median, this fraction is one-half (1/2).

Likewise, a quartile has a fraction one-fourth (1/4). The three quartiles Q1, Q2 and Q3 are such that 25 percent of the data fall below Q1, 25 percent fall between Q1 and Q2, 25 percent fall
between Q2 and Q3 and 25 percent fall above Q3. It will be seen that Q2 is the median. We can use the above formula for the calculation of quartiles as well.

The third quartile, Q3, is nine. Three-fourths (75%) of the ordered data set are less than nine. One-fourth (25%) of the ordered data set are greater than nine. The third quartile is part of the data set in this example.

For any data set:
1. The second quartile Q2 of the data set is its median.
   - the lower set: all observations that are strictly less than Q2
   - the upper set: all observations that are strictly greater than Q2.
2. The first quartile Q1 of the data set is the median of the lower set.
3. The third quartile Q3 of the data set is the median of the upper set.

The IQR can help to determine potential outliers. A value is suspected to be a potential outlier if it is less than (1.5)(IQR) below the first quartile or more than (1.5)(IQR) above the third quartile. Potential outliers always require further investigation.

The series are equally separated. These measures are the quartiles, deciles and percentiles. We call quartiles the values of a series that fall into four (4) equal parts. Three quartiles are therefore identified (Q1, Q2 and Q3) to divide the series into four equal parts.

Note: The quartile 2 (Q2) will always be equal to the median of the series.

**Example**

Calculate the quartiles of the series: {5, 2, 6, 9, 10, 13, 15}

1. The first step to be taken is the sort (ascending or descending) of the values:
2. {2, 5, 6, 9, 10, 13, 15}
3. The value that divides the above series into two equal parts is greater than 9, then the Median = Q2 = 9 that will be.
4. The steps for determining the Q1 of a set are as follows: Determine on the (adding column fi);
   - Calculate the value of (n / 4) (whether n is even or odd!);
   - To build the college column;
   - To compare the value of (n / 4) with the college’s values, starting from the first college class (the top!) And asking the question: “This college is greater than or equal to (n / 4)? “
If the answer is NO, the college spent the next class. When the answer is YES, we will stop and try the corresponding class! This will be our Class of First Quartile.

Finally, we will apply the formula for Q1, extracting the data Q1 of this class, we just found! Again the formula:

$Q1 = l_{\text{inf}} + \left[ \frac{n/4 - f_{\text{cum}}}{f_i} \right] \cdot h$

**Example**
For the set below, determine the value of the third quartile!

<table>
<thead>
<tr>
<th>Xi</th>
<th>0-10</th>
<th>10-20</th>
<th>20-30</th>
<th>30-40</th>
<th>40-50</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>fi</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>24</td>
</tr>
</tbody>
</table>

**Solution**
Step 1) We will find and calculate n (3n / 4):

Hence, we find that n = 24 and therefore (3n / 4) = 18

Step 2) builds the college:

<table>
<thead>
<tr>
<th>Xi</th>
<th>0-10</th>
<th>10-20</th>
<th>20-30</th>
<th>30-40</th>
<th>40-50</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>fi</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>6</td>
<td>3</td>
<td>24</td>
</tr>
<tr>
<td>cf</td>
<td>2</td>
<td>7</td>
<td>15</td>
<td>21</td>
<td>24</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Clues</th>
<th>&gt;18, No</th>
<th>&gt;18, No</th>
<th>&gt;18, No</th>
<th>&gt;18, Yes</th>
<th>&gt;18, Yes</th>
</tr>
</thead>
</table>

As the YES answer came in the fourth college class (30! --- 40), we will say that this will be our third class Quartile!

Step 4) we will apply the formula of Q3, using data from the Class of Q3, just identified!

$Q3 = l_{\text{inf}} + \left[ \frac{3n}{4} - f_{\text{cum}} \right] \cdot h$

$Q3 = 30 + \left[ \frac{18-15}{6} \right] \cdot 10$

Therefore, Q3 is 35
**Locations of medians**

The median is a number that measures the ‘centre’ of the data. You can think of the median as the "middle value," but it does not actually have to be one of the observed values. It is a number that separates ordered data into halves. Half the values are the same number or smaller than the median, and half the values are the same number or larger. For example, consider the following data: 1; 11.5; 6; 7.2; 4; 8; 9; 10; 6.8; 8.3; 2; 2; 10; 1.

Ordered from smallest to largest:
1; 1; 2; 2; 4; 6; 6.8; 7.2; 8; 8.3; 9; 10; 10; 11.5

Since there are 14 observations, the median is between the seventh value, 6.8, and the eighth value, 7.2. To find the median, add the two values together and divide by two.

\[
\frac{6.8 + 7.2}{2} = 7.
\]

The median is seven. Half of the values are smaller than seven and half of the values are larger than seven.

The median or second quartile is seven. The lower half of the data are 1, 1, 2, 2, 4, 6. The middle value of the lower half is 2: 1; 1; 2; 2; 4; 6.

Given an observed value \(x\) in a data set, \(x\) is the \(P^\text{th}\) percentile of the data if the percentage of the data that are less than or equal to \(x\) is \(P\). The number \(P\) is the percentile rank of \(x\).

In addition to the three quartiles, the two extreme values, the minimum \(x_{\text{min}}\) and the maximum \(x_{\text{max}}\) are also useful in describing the entire data set. Together these five numbers are called the five-number summary of the data set: \(\{x_{\text{min}}, Q_1, Q_2, Q_3, x_{\text{max}}\}\)

**The box plot**

The five-number summary is used to construct a box plot. Each of the five numbers is represented by a vertical line segment, a box is formed using the line segments at \(Q_1\) and \(Q_3\) as its two vertical sides, and two horizontal line segments are extended from the vertical segments marking \(Q_1\) and \(Q_3\) to the adjacent extreme values.

The two horizontal line segments are referred to as “whiskers,” and the diagram is sometimes called a ‘box and whisker plot.’

We caution that there are other types of box plots that differ somewhat from the ones we are constructing, although all are based on the three quartiles.
Example
Construct a box plot and find the IQR for the data 1.39, 1.76, 1.90, 2.12, 2.53, 2.71, 3.00, 3.33, 3.71, 3.88, and 4.00

Solution
\( x_{\text{min}} = 1.39, \quad Q_1 = 1.90, \quad Q_2 = 2.62, \quad Q_3 = 3.33, \quad x_{\text{max}} = 4.00 \)

The box plot is

The interquartile range is \( IQR = 3.33 - 1.90 = 1.43 \).

Example
What percentile is the value 3.33?

Solution
The data written in increasing order are: 1.39; 1.76; 1.90; 2.12; 2.53; 2.71; 3.00; 3.33; 3.71; 4.00.
The only data value that is less than or equal to 1.39 is 1.39 itself.
Since 1 is 1/10 = .10 or 10% of 10, the value 1.39 is the 10th percentile.

Eight data values are less than or equal to 3.33. Since 8 is 8/10 = .80 or 80% of 10, the value 3.33 is the 80th percentile.

Examples
1. Find the quartiles of the data set of GPAs of 1.39; 1.76; 1.90; 2.12; 2.53; 2.71; 3.00; 3.33; 3.71; 4.00.

**Solution**

This data set has \( n = 10 \) observations. Since 10 is an even number, the median is the mean of the two middle observations:

\[ x^* = \frac{(2.53 + 2.71)}{2} = 2.62. \]

Thus the second quartile is \( Q_2 = 2.62 \).

The lower and upper subsets are:

**Lower:** \( L = \{1.39, 1.76, 1.90, 2.12, 2.53\} \)

**Upper:** \( U = \{2.71, 3.00, 3.33, 3.71, 4.00\} \)

Each has an odd number of elements, so the median of each is its middle observation. Thus the first quartile is \( Q_1 = 1.90 \), the median of \( L \), and the third quartile is \( Q_3 = 3.33 \), the median of \( U \).

2. Adjoin the observation 3.88 to the data set of the previous example and find the quartiles of the new set of data.

**Solution:**

As in the previous example we first list the data in numerical order:

\[ 1.39 \ 1.76 \ 1.90 \ 2.12 \ 2.53 \ 2.71 \ 3.00 \ 3.33 \ 3.71 \ 3.88 \ 4.00 \]

This data set has 11 observations. The second quartile is its median, the middle value 2.71. Thus \( Q_2 = 2.71 \).

The lower and upper subsets are now:

**Lower:** \( L = \{1.39, 1.76, 1.90, 2.12, 2.53\} \)

**Upper:** \( U = \{3.00, 3.33, 3.71, 3.88, 4.00\} \)

The lower set \( L \) has median the middle value 1.90, so \( Q_1 = 1.90 \). The upper set has median the middle value 3.71, so \( Q_3 = 3.71 \).

It may be noted that unlike arithmetic mean, median is not affected at all by extreme values, as it is a positional average. As such, median is particularly very useful when a distribution happens to be skewed.

Another point that goes in favour of median is that it can be computed when a distribution has open-end classes.

Yet, another merit of median is that when a distribution contains qualitative data, it is the only average that can be used. No other average is suitable in case of such a distribution.
Interpreting Percentiles, Quartiles, and Median

A percentile indicates the relative standing of a data value when data are sorted into numerical order from smallest to largest.

Percentages of data values are less than or equal to the pth percentile. For example, 15% of data values are less than or equal to the 15th percentile.

- Low percentiles always correspond to lower data values.
- High percentiles always correspond to higher data values.

A percentile may or may not correspond to a value judgment about whether it is "good" or "bad." The interpretation of whether a certain percentile is "good" or "bad" depends on the context of the situation to which the data applies.

In some situations, a low percentile would be considered "good;" in other contexts a high percentile might be considered "good". In many situations, there is no value judgment that applies.

Understanding how to interpret percentiles properly is important not only when describing data, but also when calculating probabilities.

When writing the interpretation of a percentile in the context of the given data, the sentence should contain the following information.

- Information about the context of the situation being considered
- The data value (value of the variable) that represents the percentile
- The percent of individuals or items with data values below the percentile
- The percent of individuals or items with data values above the percentile.

Activities 1

1. Following are the published weights (in kg) of all of the team members of UEW basketball from the previous year.
   177; 205; 210; 210; 232; 205; 185; 185; 178; 210; 206; 212; 184; 174; 185; 242; 188; 212; 215; 247; 241; 223; 220; 260; 245; 259; 278; 270; 280; 295; 275; 285; 290; 272; 273; 280; 285; 286; 200; 215; 185; 230; 250; 241; 190; 260; 250; 302; 265; 290; 276; 228; 265.

   a. Organize the data from smallest to largest value.
   b. Find the median.
   c. Find the first quartile.
   d. Find the third quartile.

**Answer**
a). 174; 177; 184; 185; 185; 185; 188; 190; 200; 205; 205; 206; 210; 210; 210; 212; 212; 215; 215; 220; 223; 228; 230; 232; 241; 241; 242; 245; 247; 250; 250; 259; 260; 260; 265; 265; 270; 272; 273; 275; 276; 278; 280; 280; 285; 285; 286; 290; 290; 295; 302
b). 241
c). 205.5
d). 272.5

2. For the following 13 real estate prices in Ghana, calculate the IQR and determine if any prices are potential outliers. Prices are in Ghana cedis.
389,950; 230,500; 158,000; 479,000; 639,000; 114,950; 5,500,000; 387,000; 659,000; 529,000; 575,000; 488,800; 1,095,000.

Solution
Order the data from smallest to largest.
114,950; 158,000; 230,500; 387,000; 389,950; 479,000; 488,800; 529,000; 575,000; 639,000; 659,000; 1,095,000; 5,500,000

\[ M = 488,800 \]

\[ Q1 = \frac{(230,500 + 387,000)}{2} = 308,750 \]

\[ Q3 = \frac{(639,000 + 659,000)}{2} = 649,000 \]

\[ IQR = 649,000 - 308,750 = 340,250 \]

\[ (1.5)(IQR) = (1.5)(340,250) = 510,375 \]

\[ Q1 - (1.5)(IQR) = 308,750 - 510,375 = -201,625 \]

\[ Q3 + (1.5)(IQR) = 649,000 + 510,375 = 1,159,375 \]

No house price is less than \(-201,625\). However, 5,500,000 is more than 1,159,375. Therefore, 5,500,000 is a potential outlier.

3. For the two data sets in the test scores below, find:
a. The interquartile range. Compare the two interquartile ranges.
b. Any outliers in either set.

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>Q1</th>
<th>Median</th>
<th>Q3</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>Day</td>
<td>32</td>
<td>56</td>
<td>74.5</td>
<td>82.5</td>
<td>99</td>
</tr>
<tr>
<td>Night</td>
<td>25.5</td>
<td>78</td>
<td>81</td>
<td>89</td>
<td>98</td>
</tr>
</tbody>
</table>

Solution
a. The IQR for the day group is $Q_3 - Q_1 = 82.5 - 56 = 26.5$
   The IQR for the night group is $Q_3 - Q_1 = 89 - 78 = 11$

   The interquartile range (the spread or variability) for the day class is larger than the night class $IQR$. This suggests more variation will be found in the day class’s class test scores.

b. Day class outliers are found using the IQR times 1.5 rule. So,
   $Q_1 - IQR(1.5) = 56 - 26.5(1.5) = 16.25$
   $Q_3 + IQR(1.5) = 82.5 + 26.5(1.5) = 122.25$

   Since the minimum and maximum values for the day class are greater than 16.25 and less than 122.25, there are no outliers.

   Night class outliers are calculated as:
   $Q_1 - IQR (1.5) = 78 - 11(1.5) = 61.5$
   $Q_3 + IQR (1.5) = 89 + 11(1.5) = 105.5$

   For this class, any test score less than 61.5 is an outlier. Therefore, the scores of 45 and 25.5 are outliers. Since no test score is greater than 105.5, there is no upper end outlier.

4. Fifty statistics students in UEW, Winneba were asked how much sleep they get per school night (rounded to the nearest hour) and the results were collated as follows.

<table>
<thead>
<tr>
<th>Sleep per school</th>
<th>Frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
</tr>
<tr>
<td>7</td>
<td>12</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
</tr>
<tr>
<td>9</td>
<td>7</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
</tr>
</tbody>
</table>

Using the table to find:
A. The 28th percentile
B. The median
C. The third quartile (75%)
D. The 80th percentile
E. The 90th percentile
F. The 25th percentile

Solution
Compute the relative frequencies and the cumulative frequencies

<table>
<thead>
<tr>
<th>Sleep per school</th>
<th>Frequency</th>
<th>Relative frequency</th>
<th>Cumulative frequency</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2</td>
<td>0.04</td>
<td>0.04</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>0.10</td>
<td>0.14</td>
</tr>
<tr>
<td>6</td>
<td>7</td>
<td>0.14</td>
<td>0.28</td>
</tr>
<tr>
<td>7</td>
<td>12</td>
<td>0.24</td>
<td>0.52</td>
</tr>
<tr>
<td>8</td>
<td>14</td>
<td>0.28</td>
<td>0.80</td>
</tr>
<tr>
<td>9</td>
<td>7</td>
<td>0.14</td>
<td>0.94</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>0.04</td>
<td>1.00</td>
</tr>
</tbody>
</table>

A. The 28th percentile
- Notice the 0.28 in the ‘cumulative relative frequency’ column.
- 28 percent of 50 data values is 14 values.
- There are 14 values less than the 28th percentile.
- They include the two 4s, the five 5s, and the seven 6s.
- The 28th percentile is between the last six and the first seven.
- Therefore, the 28th percentile is 6.5.

B. The median
- Look again at the ‘cumulative relative frequency’ column and find 0.52.
- The median is the 50th percentile or the second quartile. 50% of 50 is 25.
- There are 25 values less than the median, namely the two 4s, the five 5s, the seven 6s, and eleven of the 7s.
- The median or 50th percentile is between the 25th, or seven, and 26th, or seven, values.
- Therefore, the median is seven

C. The third quartile (75%)
- The third quartile is the same as the 75th percentile.
- We can ‘eyeball’ this answer if you look at the ‘cumulative relative frequency’ column, and find 0.52 and 0.80.
- When we have all the fours, fives, sixes and sevens, we have 52% of the data
- When we include all the 8s, we have 80% of the data
Therefore, the 75th percentile, then, must be an eight (8)

Alternatively
- Another way to look at the problem is to find 75% of 50, which is 37.5, and round up to 38.
- The third quartile, Q3, is the 38th value, which is an eight.
- We can check this answer by counting the values.
- There are 37 values below the third quartile and 12 values above.

D. The 80th percentile
- The 80th percentile is between the last eight and the first nine in the table (between the 40th and 41st values).
- Therefore, we need to take the mean of the 40th and 41st values. The 80th percentile = (8 + 9)/2 = 8.5

E. The 90th percentile
- The 90th percentile will be the 45th data value (location is 0.90(50) = 45) and the 45th data value is nine (9).

F. The 25th percentile
- Q1 is also the 25th percentile. The 25th percentile location calculation: \( P_{25} = 0.25(50) = 12.5 \approx 13 \) (the 13th data value).
- Thus, the 25th percentile is six.

5. On a timed math test, the first quartile for time it took to finish the exam was 35 minutes. Interpret the first quartile in the context of this situation.

Solution
- Twenty-five percent of students finished the exam in 35 minutes or less.
- Seventy-five percent of students finished the exam in 35 minutes or more.
- A low percentile could be considered good, as finishing more quickly on a timed exam is desirable. (If you take too long, you might not be able to finish.)

6. On a 20 question math test, the 70th percentile for number of correct answers was 16. Interpret the 70th percentile in the context of this situation.

Solution
- Seventy percent of students answered 16 or fewer questions correctly.
- Thirty percent of students answered 16 or more questions correctly.
• A higher percentile could be considered good, as answering more questions correctly is desirable.

1. At a UEW, Winneba distance campus, it was found that the 30th percentile of credit units that students are enrolled for is seven units. Interpret the 30th percentile in the context of this situation.

Solution
• Thirty percent of students are enrolled in seven or fewer credit units.
• Seventy percent of students are enrolled in seven or more credit units.
• In this example, there is no ‘good’ or ‘bad’ value judgment associated with a higher or lower percentile.
• Therefore, students attend the campus for varied reasons and needs, and their course load varies according to their needs.

8. HPERS Department of UEW, Winneba is applying for a grant that will be used to add fitness equipment to the gym. The Head of Department surveyed 15 anonymous students to determine how many minutes a day the students spend exercising in the gym. The results from the 15 anonymous students are shown as follows:
   a. 0 minutes; 40 minutes; 60 minutes; 30 minutes; 60 minutes
   b. 10 minutes; 45 minutes; 30 minutes; 300 minutes; 90 minutes;
   c. 30 minutes; 120 minutes; 60 minutes; 0 minutes; 20 minutes

Use the information to determine the following five values:
   a. The minimum time spent
   b. The first quartile
   c. The median time
   d. The third quartile
   e. The maximum time spent
   f. If you were the HOD of HPERS, how would you be justified in purchasing the new fitness equipment?

Solutions
   a. Min = 0
   b. \( Q1 = 20 \)
   c. Med = 40
   d. \( Q3 = 60 \)
   e. Max = 300

f. Since 75% of the students exercise for 60 minutes or less daily, and since the \( IQR \) is 40 minutes \((60 – 20 = 40)\), we know that half of the students surveyed exercise between 20 minutes and 60 minutes daily.
This seems a reasonable amount of time spent exercising, so the principal would be justified in purchasing the new equipment.

However, the HOD needs to be careful. The value 300 appears to be a potential outlier. 

\[ Q3 + 1.5(IQR) = 60 + (1.5)(40) = 120. \]

The value 300 is greater than 120 so it is a potential outlier. If we delete it and calculate the five values, we get the following values:

- Min = 0
- \( Q1 = 20 \)
- \( Q3 = 60 \)
- Max = 120
- We still have 75% of the students exercising for 60 minutes or less daily and half of the students exercising between 20 and 60 minutes a day.
- However, 15 students is a small sample and the HOD should survey more students to be sure of his survey results.

**Locations of Skewness: measures of asymmetry**

It may be that frequency distributions differ in three ways: Average value, Variability or dispersion, and Shape. Here, our main spotlight will be on the shape of frequency distribution.

Generally, there are two comparable characteristics called skewness and kurtosis that help us to understand a distribution.

Two distributions may have the same mean and standard deviation but may differ widely in their overall appearance as can be seen from the following:
In both these distributions the value of mean and standard deviation is the same ($X = 15, \sigma = 5$). But it does not imply that the distributions are alike in nature.

The distribution on the left-hand side is a symmetrical one whereas the distribution on the right-hand side is symmetrical or skewed. Measures of skewness help us to distinguish between different types of distributions.

Some important definitions of skewness are as follows:

1. When a series is not symmetrical it is said to be asymmetrical or skewed (Croxton & Cowden).
2. Skewness refers to the asymmetry or lack of symmetry in the shape of a frequency distribution (Morris Hamburg).
3. Measures of skewness tell us the direction and the extent of skewness. In symmetrical distribution the mean, median and mode are identical. The more the mean moves away from the mode, the larger the asymmetry or skewness (Simpson & Kalka).
4. A distribution is said to be ‘skewed’ when the mean and the median fall at different points in the distribution, and the balance (or centre of gravity) is shifted to one side or the other-to left or right (Garrett).

The above definitions show that the term ‘skewness’ refers to lack of symmetry i.e., when a distribution is not symmetrical (or is asymmetrical) it is called a skewed distribution.

The concept of skewness will be clear from the following three diagrams showing a symmetrical distribution, a positively skewed distribution and a negatively skewed distribution.
1. Symmetrical Distribution

It is clear from the diagram that in a symmetrical distribution the values of mean, median and mode coincide. The spread of the frequencies is the same on both sides of the centre point of the curve.

2. Asymmetrical Distribution

A distribution, which is not symmetrical, is called a skewed distribution and such a distribution could either be positively skewed or negatively skewed as would be clear from the diagrams (b) and (c).

3. Positively Skewed Distribution

In the positively skewed distribution the value of the mean is maximum and that of mode least-the median lies in between the two as is clear from the diagram (b).

4. Negatively Skewed Distribution

The following is the shape of negatively skewed distribution. In a negatively skewed distribution the value of mode is maximum and that of mean least-the median lies in between the two. In the
positively skewed distribution the frequencies are spread out over a greater range of values on the high-value end of the curve (the right-hand side) than they are on the low-value end. In the negatively skewed distribution the position is reversed, i.e. the excess tail is on the left-hand side.

It should be noted that in moderately symmetrical distributions the interval between the mean and the median is approximately one-third of the interval between the mean and the mode. It is this relationship, which provides a means of measuring the degree of skewness.

Tests of Skewness
In order to ascertain whether a distribution is skewed or not the following tests may be applied. Skewness is present if:

1. The values of mean, median and mode do not coincide.
2. When the data are plotted on a graph they do not give the normal bell-shaped form i.e. when cut along a vertical line through the centre the two halves are not equal.
3. The sum of the positive deviations from the median is not equal to the sum of the negative deviations.
4. Quartiles are not equidistant from the median.
5. Frequencies are not equally distributed at points of equal deviation from the mode.

On the contrary, when skewness is absent, i.e. in case of a symmetrical distribution, the following conditions are satisfied:

1. The values of mean, median and mode coincide.
2. Data when plotted on a graph give the normal bell-shaped form.
3. Sum of the positive deviations from the median is equal to the sum of the negative deviations.
4. Quartiles are equidistant from the median.
5. Frequencies are equally distributed at points of equal deviations from the mode.

Measures of Skewness
It is an indicator of the shape of the data distribution

1. AS = 0 → distribution is symmetrical;
2. AS > 0 → positive distribution is asymmetric;

3. AS < 0 → distribution is asymmetrical negative.

There are four measures of skewness, each divided into absolute and relative measures. The relative measure is known as the coefficient of skewness and is more frequently used than the absolute measure of skewness.

Further, when a comparison between two or more distributions is involved, it is the relative measure of skewness, which is used. The relative measures of skewness are:
(i) Karl Pearson's measure  
(ii) Bowley’s measure  
(iii) Kelly’s measure  
(iv) Moment’s measure.

**Karl Pearson’s measure of skewness**

The formula for measuring skewness as given by Karl Pearson is as follows:

\[
\text{Skewness} = \text{Mean} - \text{Mode}
\]

Coefficient of skewness is

\[
sk = \frac{\text{mean} - \text{mode}}{\sigma}
\]

In case the mode is indeterminate, the coefficient of skewness is:

\[
sk = \frac{\text{mean} - 3(\text{median} - 2\text{mean})}{\sigma} = \frac{3(\text{mean} - \text{median})}{\sigma}
\]

\[
sk = \frac{3(\bar{x} - M)}{s}
\]

Since \(\frac{3(\text{mean} - \text{mode})}{\sigma} = \frac{\text{mean} - \text{mode}}{\sigma}\)

Then we can obtain the following variants:

- \(3\text{Mean} - 3\text{Median} = \text{Mean} - \text{Mode}\)
- \(\text{Mode} = \text{Mean} - 3\text{Mean} + 3\text{Median}\)
- \(\text{Mode} = 3\text{Median} - 2\text{Mean}\)

Thus, the direction of skewness is determined by ascertaining whether the mean is greater than the mode or less than the mode. If it is greater than the mode, then skewness is positive. But when the mean is less than the mode, it is negative.

The difference between the mean and mode indicates the extent of departure from symmetry. It is measured in standard deviation units, which provide a measure independent of the unit of measurement.

The value of coefficient of skewness is zero, when the distribution is symmetrical. Normally, this coefficient of skewness lies between +1.

If the mean is greater than the mode, then the coefficient of skewness will be positive, otherwise negative.
Example
1. Given the following data, calculate the Karl Pearson's coefficient of skewness: \( \Sigma x = 452; \Sigma x^2=24270; \text{Mode} = 43.7 \) and \( N = 10. \)

Solution

Pearson’s coefficient is \( \frac{3(x - M)}{s} \)

\[
x = \frac{\sum x}{n} = \frac{452}{10} = 45.2
\]

\[
s = \sqrt{\frac{\sum x^2}{n} - \left(\frac{\sum x}{n}\right)^2} = \sqrt{\frac{24270}{10} - \left(\frac{452}{10}\right)^2} = 19.59
\]

Applying the values of mean, mode and standard deviation in the above formula,

\[
sk = \frac{(x - M)}{s} = \frac{45.2 - 43.7}{19.59} = 0.08
\]

This shows that there is a positive skewness though the extent of skewness is marginal.

2. From the following data, calculate the measure of skewness using the mean, median and standard deviation:

<table>
<thead>
<tr>
<th>( x )</th>
<th>10 - 20</th>
<th>20 - 30</th>
<th>30 - 40</th>
<th>40 - 50</th>
<th>50 - 60</th>
<th>60 - 70</th>
<th>70 - 80</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
<td>18</td>
<td>30</td>
<td>40</td>
<td>55</td>
<td>38</td>
<td>20</td>
<td>16</td>
</tr>
</tbody>
</table>

Solution

Using \( A = \text{Assumed mean} = 45, \ cf = \text{Cumulative frequency}, \ d = \text{Deviation from assumed mean}, \) and \( i = 10 \) to find the mean:

<table>
<thead>
<tr>
<th>interval</th>
<th>( f )</th>
<th>( x )</th>
<th>( d = x - a )</th>
<th>( fd )</th>
<th>( fd^2 )</th>
<th>( cf )</th>
</tr>
</thead>
<tbody>
<tr>
<td>10 - 20</td>
<td>18</td>
<td>15</td>
<td>-3</td>
<td>-54</td>
<td>162</td>
<td>18</td>
</tr>
<tr>
<td>20 - 30</td>
<td>30</td>
<td>25</td>
<td>-2</td>
<td>-60</td>
<td>120</td>
<td>48</td>
</tr>
</tbody>
</table>
The result shows that the distribution is negatively skewed, but the extent of skewness is extremely negligible.

**Bowley’s measure of skewness**

Bowley developed a measure of skewness, which is based on quartile values. The formula for measuring skewness is: $$Skb = \frac{Q_3 + Q_1 - 2M}{Q_3 - Q_1}.$$  

Where $Q_3$ and $Q_1$ are upper and lower quartiles and $M$ is the median. The value of this skewness varies between $+1$. In the case of open-ended distribution as well as where extreme values are found in the series, this measure is particularly useful.

In a symmetrical distribution, skewness is zero. This means that $Q_3$ and $Q_1$ are positioned equidistantly from $Q_2$ that is, the median.
In symbols, \( Q_3 - Q_2 = Q_2 - Q_1' \)

In contrast, when the distribution is skewed, then \( Q_3 - Q_2 \) will be different from \( Q_2 - Q_1' \). When \( Q_3 - Q_2 \) exceeds \( Q_2 - Q_1' \) then skewness is positive.

As against this; when \( Q_3 - Q_2 \) is less than \( Q_2 - Q_1' \) then skewness is negative.

Bowley's measure of skewness can be written as: Skewness = \( (Q_3 - Q_2) - (Q_2 - Q_1) \) or \( Q_3 - Q_2 = Q_2 - Q_1 \) or \( Q_3 + Q_1 - 2Q_2 \) (2Q2 is 2M).

However, this is an absolute measure of skewness. As such, it cannot be used while comparing two distributions where the units of measurement are different. In view of this limitation, Bowley suggested a relative measure of skewness is \( SkB = \frac{Q_3 + Q_1 - 2M}{Q_3 - Q_1} \).

**Example**

1. For a distribution, Bowley’s coefficient of skewness is - 0.56, \( QI = 16.4 \) and Median = 24.2. What is the coefficient of quartile deviation?

**Solution:**

Bowley's coefficient of skewness is \( SkB = \frac{Q_3 + Q_1 - 2M}{Q_3 - Q_1} \)

Substituting the values in the above formula,

\[
SkB = \frac{Q_3 + Q_1 - 2M}{Q_3 - Q_1} \Leftrightarrow -0.56 = \frac{Q_3 + 16.4 - 2(2 \times 24.2)}{Q_3 - 16.4}
\]

Finding Q3, we will obtain:

\[
\frac{Q_3 + 16.4 - 2(2 \times 24.2)}{Q_3 - 16.4} \Leftrightarrow Q_3 = \frac{41.184}{1.56} = 26.4
\]

Now, we have the values of both the upper and the lower quartiles.

Coefficient of quartile deviation is \( C_q = \frac{Q_3 - Q_1}{Q_3 + Q_1} = \frac{26.4 - 16.4}{26.4 + 16.4} = 0.234 \)

2. Calculate an appropriate measure of skewness from the following data:

<table>
<thead>
<tr>
<th>Values (₵)</th>
<th>&lt;50</th>
<th>50-100</th>
<th>100-150</th>
<th>150-200</th>
<th>200+</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>40</td>
<td>80</td>
<td>130</td>
<td>60</td>
<td>30</td>
</tr>
</tbody>
</table>

**Solution**

It should be noted that the series given in the question is an open-ended series. As such, Bowley's coefficient of skewness, which is based on quartiles, would be the most appropriate measure of skewness in this case.
In order to calculate the quartiles and the median, we have to use the cumulative frequency.

The table is reproduced below with the cumulative frequency.

<table>
<thead>
<tr>
<th>Values (₵)</th>
<th>&lt;50</th>
<th>50-100</th>
<th>100-150</th>
<th>150-200</th>
<th>200+</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency</td>
<td>40</td>
<td>80</td>
<td>130</td>
<td>60</td>
<td>30</td>
</tr>
<tr>
<td>Cumulative frequency</td>
<td>40</td>
<td>120</td>
<td>250</td>
<td>310</td>
<td>340</td>
</tr>
</tbody>
</table>

The quartile is

\[ Q = L + \frac{L_2 - L_1}{f_1}(m-c) \]

The lower quartile, Q1 is

\[ M_1 = \frac{(n+1)}{4}th = \frac{1 \times 341}{4}th = 85.25th, \] which lies in 50 - 100 class.

Therefore, \( Q_1 = 50 + \frac{100 - 50}{80}(85.25 - 40) = 78.28 \)

The median class is

\[ M = \frac{2(n+1)}{4}th = \frac{341}{2}th = 170.50th, \] which lies in 100-150 class

Therefore, the median itself is

\[ M = 100 + \frac{150 - 100}{130}(170.50 - 120) = 119.40 \]

The upper quartile, Q3 is

\[ M_3 = \frac{3(n+1)}{4}th = \frac{3 \times 341}{4}th = 255.75th, \] which lies in 150 - 200 class.

\[ Q_1 = 150 + \frac{200 - 150}{60}(255.75 - 250) = 154.79 \]

Bowley's coefficient of skewness is

\[ sk = \frac{Q_3 + Q_1 - 2M}{Q_3 - Q_1} = \frac{154.75 + 78.28 - 2 \times 119.40}{154.75 - 78.28} = -0.075 \]

This shows that there is a negative skewness, which has a very negligible magnitude.

**Kelly’s measure of skewness**

Kelly developed another measure of skewness, which is based on percentiles and deciles. The formula for measuring skewness is as follows:

Coefficient of skewness is

\[ sk = \frac{P_{90} - 2P_{50} + P_{10}}{P_{90} - P_{10}}, \] or

\[ sk = \frac{D_1 + D_9 - 2M}{D_9 - D_1}, \] where P and D stand for percentile and decile respectively. In order to calculate the coefficient of skewness by this formula, we have to ascertain the values of 10th, 50th and 90th percentiles. Somehow, this measure of skewness is seldom used.

**Example**

Use Kelly’s measure to calculate skewness from the table below:

<table>
<thead>
<tr>
<th>Intervals</th>
<th>10 - 20</th>
<th>20-30</th>
<th>30-40</th>
<th>40-50</th>
<th>550-60</th>
<th>60-70</th>
<th>70-80</th>
</tr>
</thead>
<tbody>
<tr>
<td>f</td>
<td>18</td>
<td>30</td>
<td>40</td>
<td>55</td>
<td>38</td>
<td>20</td>
<td>16</td>
</tr>
</tbody>
</table>
Solution

Now we have to calculate $P_{10}$, $P_{50}$ and $P_{90}$.

$$m_p = \frac{p(n + 1)h}{100}$$

$$m_{10} = \frac{10(n + 1)h}{100} = \frac{217 + 1}{10} = 21.8h$$, which lies in the 20 - 30 class.

Therefore, $P_{10} = 20 + \frac{30}{30}(21.8 - 18) = 21.27$

$$m_{50} = \frac{50(n + 1)h}{100} = \frac{217 + 1}{2} = 109h$$, which lies in the class 40 - 50.

Therefore, $P_{50} = 40 + \frac{50}{55}(109 - 88) = 43.82$

$$m_{90} = \frac{90(n + 1)h}{100} = \frac{217 + 1}{2} = 196.2h$$, which lies in the class 60 - 70.

Therefore, $P_{90} = 60 + \frac{70}{20}(196.20 - 181) = 67.60$

Hence, Kelley’s skewness, Skk is

$$Skk = \frac{P_{90} - 2P_{50} + P_{10}}{P_{90} - P_{10}} = \frac{67.60 - (2 \times 43.82) + 21.27}{67 - 60 - 21.27} = 0.027$$

This shows that the series is positively skewed though the extent of skewness is extremely negligible. It may be recalled that if there is a perfectly symmetrical distribution, then the skewness will be zero. One can see that the above answer is very close to zero.

Moment’s measure of skewness

In mechanics, the term moment is used to denote the rotating effect of a force. In Statistics, it is used to indicate peculiarities of a frequency distribution. The utility of moments lies in the sense that they indicate different aspects of a given distribution.
Thus, by using moments, we can measure the central tendency of a series, dispersion or variability, skewness and the *peakchedness* of the curve. The moments about the actual arithmetic mean are denoted by \( \mu \).

In statistical theory, location and variability are referred to as the first and second *moments* of a distribution. The third and fourth moments are called *skewness* and *kurtosis*. Skewness refers to whether the data is skewed to larger or smaller values and kurtosis indicates the propensity of the data to have extreme values. Generally, metrics are not used to measure skewness and kurtosis; instead, these are discovered through visual displays.

The kurtosis index measures the extent to which the peak of a unimodal frequency distribution departs from the shape of normal distribution. A value of zero corresponds to a normal distribution; positive values indicate a distribution that is more pointed than a normal distribution and a negative value a flatter distribution.

The first four *moments* about zero in relation to individual items are as follows:

First moment is \( \bar{x}_1 = \frac{1}{n} \sum x_i \)

Second moment is \( \bar{x}_2 = \frac{1}{n} \sum (x_i)^2 \)

Third moment is \( \bar{x}_3 = \frac{1}{n} \sum (x_i)^3 \)

Fourth moment is \( \bar{x}_4 = \frac{1}{n} \sum (x_i)^4 \)

The first four moments about mean or *central moments* in relation to individual items are:

First moment is \( \mu_1 = \frac{1}{n} \sum (x_i - \bar{x}) \)

Second moment is \( \mu_2 = \frac{1}{n} \sum (x_i - \bar{x})^2 \)

Third moment is \( \mu_3 = \frac{1}{n} \sum (x_i - \bar{x})^3 \)

Fourth moment is \( \mu_4 = \frac{1}{n} \sum (x_i - \bar{x})^4 \)

The first four moments about mean or *central moments* in the case of a frequency distribution are as follows:

First moment is \( \mu_i = \frac{1}{n} \sum f_i (x_i - \bar{x}) \)

Second moment is \( \mu_2 = \frac{1}{n} \sum f_i (x_i - \bar{x})^2 \)
Third moment is \( \mu_3 = \frac{1}{n} \sum f_i (x_i - \bar{x})^3 \)

Fourth moment is \( \mu_4 = \frac{1}{n} \sum f_i (x_i - \bar{x})^4 \)

It may be noted that:
- the first central moment is zero, that is, \( \mu_1 = 0 \)
- The second central moment is \( \mu_2 = \sigma^2 \), indicating the variance.
- The third central moment \( \mu_3 \) is used to measure skewness.
- The fourth central moment \( \mu_4 \) gives an idea about the Kurtosis.

Karl Pearson measure of skewness is based on the third and second central moments as \( \beta_1 = \frac{\mu_3}{\mu_2^{2/3}} \)

**Example**
Find the first, second, third and fourth moments about zero for the set of numbers 2,3,4,5 and 6.

**Solution**
a). First moment is \( \bar{x}_1 = \frac{1}{n} \sum (x_i) = \frac{2 + 3 + 4 + 5 + 6}{5} = 4 \)
b). Second moment is \( \bar{x}_2 = \frac{1}{n} \sum (x_i)^2 = \frac{4 + 9 + 16 + 25 + 36}{5} = 18 \)
c). Third moment is \( \bar{x}_3 = \frac{1}{n} \sum (x_i)^3 = \frac{8 + 27 + 64 + 125 + 216}{5} = 88 \)
d). Fourth moment is \( \bar{x}_4 = \frac{1}{n} \sum (x_i)^4 = \frac{16 + 81 + 256 + 625 + 1296}{5} = 454.80 \)

2. Using the five figures 2,3,4,5 and 6, find the first, second, third and fourth moments about the mean.

**Solutions**
a). First moment is \( \mu_1 = \frac{1}{n} \sum (x - \bar{x}) = \frac{(2 - 4) + (3 - 4) + (4 - 4) + (5 - 4) + (6 - 4)}{5} = 0 \)
b). Second moment is \( \mu_2 = \frac{1}{n} \sum (x - \bar{x})^2 = \frac{(2 - 4)^2 + (3 - 4)^2 + (4 - 4)^2 + (5 - 4)^2 + (6 - 4)^2}{5} = 2 \)
c). Third moment is \( \mu_3 = \frac{1}{n} \sum (x - \bar{x})^3 = \frac{(2 - 4)^3 + (3 - 4)^3 + (4 - 4)^3 + (5 - 4)^3 + (6 - 4)^3}{5} = 0 \)
d). Fourth moment is
\[
\mu_4 = \frac{1}{n} \sum (x - \bar{x})^4 = \frac{(2 - 4)^4 + (3 - 4)^4 + (4 - 4)^4 + (5 - 4)^4 + (6 - 4)^4}{5} = 6.8
\]

3. Calculate the first four central moments from the following data:

<table>
<thead>
<tr>
<th>Interval</th>
<th>50-60</th>
<th>60-70</th>
<th>70-80</th>
<th>80-90</th>
<th>90-100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>5</td>
<td>12</td>
<td>20</td>
<td>7</td>
<td>6</td>
</tr>
</tbody>
</table>

**Solution**

<table>
<thead>
<tr>
<th>Interval</th>
<th>f</th>
<th>x</th>
<th>d=x-A</th>
<th>u = d/c, c = 10</th>
<th>fu</th>
<th>fu^2</th>
<th>fu^3</th>
<th>fu^4</th>
</tr>
</thead>
<tbody>
<tr>
<td>50-60</td>
<td>5</td>
<td>55</td>
<td>-20</td>
<td>-2</td>
<td>-10</td>
<td>20</td>
<td>-40</td>
<td>80</td>
</tr>
<tr>
<td>600-70</td>
<td>12</td>
<td>65</td>
<td>-10</td>
<td>-1</td>
<td>-12</td>
<td>12</td>
<td>-12</td>
<td>12</td>
</tr>
<tr>
<td>70-80</td>
<td>20</td>
<td>A = 75</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>80-90</td>
<td>7</td>
<td>85</td>
<td>10</td>
<td>1</td>
<td>7</td>
<td>7</td>
<td>7</td>
<td>7</td>
</tr>
<tr>
<td>90-100</td>
<td>6</td>
<td>95</td>
<td>20</td>
<td>2</td>
<td>12</td>
<td>24</td>
<td>48</td>
<td>96</td>
</tr>
<tr>
<td>Total</td>
<td>50</td>
<td>0</td>
<td>0</td>
<td>-3</td>
<td>63</td>
<td>-4</td>
<td>195</td>
<td></td>
</tr>
</tbody>
</table>

\[
\mu_1' = \frac{1}{n} \sum fu \times c = \frac{-3 \times 10}{50} = -0.6
\]

\[
\mu_2' = \frac{1}{n} \sum fu^2 \times c = \frac{63 \times 10}{50} = 12.60
\]

\[
\mu_3' = \frac{1}{n} \sum fu^3 \times c = \frac{-4 \times 10}{50} = -0.8
\]

\[
\mu_4' = \frac{1}{n} \sum fu^4 \times c = \frac{195 \times 10}{50} = 19
\]

Moments about Mean:

\[
\mu_1 = \mu_1' - \mu_1 = -0.6 - (-0.6) = 0
\]

\[
\mu_2 = \mu_2' - \mu_1^2 = 10 - (-0.6)^2 = 10 - 0.36 = 9.64
\]
\[ \mu_3 = \mu_1' - 3 \mu_2 \mu_1' + 2 \mu_1^{\prime 3} = -0.8 - 3(12.6)(-0.6) + 2(-0.6)^3 = -0.8 + 22.68 + 0.432 = 22.312 \]

\[ \mu_4 = \mu_2' - 4 \mu_3 \mu_2' + 6 \mu_2 \mu_1' - 3 \mu_1^{\prime 4} = 19 + 4(-0.8)(-0.6) + 6(10)(-0.6)^2 - 3(-0.6)^4 = 19 + 1.92 + 21.60 - 0.3888 = 42.1312 \]

**Measures of kurtosis**

![Diagram showing types of curves: Leptokurtic, Mesokurtic, Platykurtic](image)

**Types of Curves**

Kurtosis measures how fat or thin the tails of a distribution are relative to a normal distribution. Distributions with long tails are called leptokurtic, and distributions with short tails are called platykurtic. Normal distributions have zero kurtosis.

Distributions also differ from each other in terms of how large or “fat” their tails. If the upper distribution has relatively more scores in its tails; its shape is called leptokurtic, and if the lower distribution has relatively fewer scores in its tails; its shape is called platykurtic.

As distributions differ in kurtosis, the top distribution having long tails is called “leptokurtic”, and the bottom distribution has short tails is called “platykurtic”.

Kurtosis is another measure of the shape of a frequency curve. It is a Greek word, which means bulginess. While skewness signifies the extent of asymmetry, kurtosis measures the degree of peakedness of a frequency distribution. Karl Pearson classified curves into three types on the basis of the shape of their peaks. These are mesokurtic, leptokurtic and platykurtic.
It can be seen that mesokurtic curve is neither too much flattened nor too much peaked. In fact, this is the frequency curve of a normal distribution. Leptokurtic curve is a more peaked than the normal curve. In contrast, platykurtic is a relatively flat curve.

The coefficient of kurtosis as given by Karl Pearson is \( \beta_2 = \frac{\mu_4}{\mu_2^2} \). In case of a normal distribution, that is, mesokurtic curve, the value of \( \beta_2 = 3 \). If \( \beta_2 \) turn out to be \( \beta_2 > 3 \), the curve is called a leptokurtic curve and is more peaked than the normal curve. Again, when \( \beta_2 < 3 \), the curve is called a platykurtic curve and is less peaked than the normal curve.

Another measure of kurtosis is based on both quartiles and percentiles and is given by the following formula: 
\[
Ku = \frac{Q}{P_{90} - P_{10}},
\]
where \( K = \text{kurtosis} \), \( Q = 1/2(Q_3 - Q_1) \) is the semi-interquartile range; \( P_{90} \) is 90\(^{th}\) percentile and \( P_{10} \) is the 10th percentile. This is also known as the percentile coefficient of kurtosis. In case of the normal distribution, the value of \( K \) is 0.263.

The measure of kurtosis, \( Ku = \sum \left(\frac{x - \mu}{\sigma}\right)^4 - 3 \) is similar to the definition of skew. The value ‘3’ is subtracted to define “no kurtosis” as the kurtosis of a normal distribution. Otherwise, a normal distribution would have a kurtosis of 3.

Using percentiles and quartiles, the degree of flattening of the distribution, is an indicator of the shape of this distribution. The coefficient of kurtosis, is calculated as 
\[
C = \frac{\left(Q_3 - Q_1\right)}{2(P_{90} - P_{10})}.
\]

The fourth moment of the standard normal random variable is 3, and the kurtosis compares the fourth moment of the standardized random variable to this value 
\[
Ku = E\left(\frac{x - \mu}{\sigma}\right)^4 - 3.
\]
Random variables with a negative kurtosis are called leptokurtic. Lepto means slender. Random variables with a positive kurtosis are called platykurtic. Platy means broad.

- **Leptokurtic**: when the distribution has a frequency curve rather closed, with the data strongly concentrated around its center, \( C < 0.263 \)
- **Mesokurtic**: when data is fairly concentrated around its center, \( C = 0.263 \).
- **Platykurtic**: when the distribution has a frequency curve more open with data weakly concentrated around its center, \( C > 0.263 \).
The measure of kurtosis is very helpful in the selection of an appropriate average. For example, for normal distribution, mean is most appropriate; for a leptokurtic distribution, median is most appropriate; and for platykurtic distribution, the quartile range is most appropriate.

**Example**

From the data given below, calculate the percentile coefficient of kurtosis.

<table>
<thead>
<tr>
<th>Wage</th>
<th>50-60</th>
<th>60-70</th>
<th>70-80</th>
<th>80-90</th>
<th>90-100</th>
<th>100-110</th>
<th>110-120</th>
</tr>
</thead>
<tbody>
<tr>
<td>Workers</td>
<td>10</td>
<td>14</td>
<td>18</td>
<td>24</td>
<td>16</td>
<td>12</td>
<td>6</td>
</tr>
</tbody>
</table>

**Solution**

Let us compute the cumulative frequencies of the data set as follows:

<table>
<thead>
<tr>
<th>Wage</th>
<th>50-60</th>
<th>60-70</th>
<th>70-80</th>
<th>80-90</th>
<th>90-100</th>
<th>100-110</th>
<th>110-120</th>
</tr>
</thead>
<tbody>
<tr>
<td>Workers</td>
<td>10</td>
<td>14</td>
<td>18</td>
<td>24</td>
<td>16</td>
<td>12</td>
<td>6</td>
</tr>
<tr>
<td>cf</td>
<td>10</td>
<td>24</td>
<td>42</td>
<td>66</td>
<td>82</td>
<td>94</td>
<td>100</td>
</tr>
</tbody>
</table>

\[
m_1 = \frac{1(n + 1)}{4} \quad th = \frac{1(100 + 1)}{4} \quad th = 25.25th, \text{ which lies in the 70-80 class}
\]

\[
Q_1 = L_1 + \frac{L_2 - L_1}{f_1} (m - c) = 70 + \frac{80 - 70}{18} (25.25 - 24) = 70.69
\]

\[
m_3 = \frac{3(n + 1)}{4} th = \frac{3(100 + 1)}{4} th = 75.75th, \text{ which falls in 90 - 100 class interval}
\]

\[
Q_3 = L_1 + \frac{L_2 - L_1}{f_1} (m - c) = 90 + \frac{100 - 90}{16} (75.75 - 66) = 96.09
\]

\[
P_{10} = \frac{10(n + 1)th}{100} = \frac{(100 + 1)}{10} th = 10.10th. \text{ This falls in 60 - 70 class interval}
\]

\[
P_{10} = L_1 + \frac{L_2 - L_1}{f_1} (m - c) = 60 + \frac{70 - 60}{14} (10.10 - 10) = 60.07
\]

\[
P_{90} = \frac{90(n + 1)th}{100} = \frac{9(100 + 1)}{10} th = 90.90th \text{ This falls in 100 - 110 class interval}
\]

\[
P_{90} = L_1 + \frac{L_2 - L_1}{f_1} (m - c) = 100 + \frac{110 - 100}{12} (90.90 - 82) = 107.41
\]

Now, \[Ku = \frac{(Q_3 - Q_1)}{2(P_{90} - P_{10})} = \frac{(96.09 - 70.69)}{2(1007.41 - 60.07)} = 0.268\]

It will be seen that the distribution is very close to normal distribution as the value of K is 0.268, which is extremely close to 0.263.
Exercises
1. Forty randomly selected students were asked the number of pairs of sneakers they owned. Let \( X = \) the number of pairs of sneakers owned. The results are as follows:

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f )</td>
<td>2</td>
<td>5</td>
<td>8</td>
<td>12</td>
<td>12</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

a. Find the first quartile.
b. Find the median.
c. Find the third quartile.
d. Find the 40th percentile.
e. Find the 90th percentile.

2. The mean score on a standardized mathematics examination is 49.6 and the standard deviation is 1.35. Dede is told that the \( z \)-score of his exam score is \(-1.19\).

a. Is Dede’s score above average or below average?
b. What was Dede’s actual score on the examination?

3. Calculate Karl Pearson’s coefficient of skewness from the following data:

<table>
<thead>
<tr>
<th>Sales</th>
<th>10-12</th>
<th>12-14</th>
<th>14-16</th>
<th>16-18</th>
<th>18-20</th>
<th>20-22</th>
<th>22-24</th>
<th>24-26</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>12</td>
<td>18</td>
<td>35</td>
<td>42</td>
<td>50</td>
<td>30</td>
<td>16</td>
<td>8</td>
</tr>
</tbody>
</table>

4. Calculate the first four moments about the mean from the following data. Also calculate the values of \( \beta_1 \) and \( \beta_2 \)

<table>
<thead>
<tr>
<th>Marks</th>
<th>0-10</th>
<th>10-20</th>
<th>20-30</th>
<th>30-40</th>
<th>40-50</th>
<th>50-60</th>
<th>60-70</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>5</td>
<td>12</td>
<td>18</td>
<td>40</td>
<td>15</td>
<td>7</td>
<td>3</td>
</tr>
</tbody>
</table>

Summary
The average value cannot adequately describe a set of observations, unless all the observations are the same. It is necessary to describe the variability or dispersion of the observations. Again, in two or more distributions the central value may be the same but still there can be wide disparities in the formation of distribution.
Furthermore, two distributions may have the same mean and standard deviation but may differ widely in their overall appearance in terms of symmetry and skewness. To distinguish between different types of distributions, we may use:

- Measures of skewness and kurtosis
- The percentile rank and \( z \)-score of a measurement
- The three quartiles
- The five-number summary in the box plot

We therefore need the measures of dispersion to determine the differences and similarities of data sets and make the necessary judgements.
Introduction
The terms variability, spread, and dispersion are synonyms, and refer to how spread out a distribution is. There are four frequently used measures of variability, namely range, interquartile range, variance, and standard deviation.

An important characteristic of any set of data is the variation in the data. In some data sets, the data values are concentrated closely near the mean; in other data sets, the data values are more widely spread out from the mean. The most common measure of variation, or spread, is the standard deviation. The standard deviation is a number that measures how far data values are from their mean.

The various measures of central value give us one single figure that represents the entire data. But the average alone cannot adequately describe a set of observations, unless all the observations are the same. It is necessary to describe the variability or dispersion of the observations. In two or more distributions the central value may be the same but still there can be wide disparities in the formation of distribution. Measures of dispersion help us in studying this important characteristic of a distribution.

Our learning indicators are:
1. Determine the relative variability of two distributions
2. Compute the range
3. Compute the inter-quartile range
4. Compute the variance in the population
5. Estimate the variance from a sample
6. Compute the standard deviation from the variance
Some important definitions of dispersion
1. Dispersion is the measure of the variation of the items (A.L. Bowley)
2. The degree to which numerical data tend to spread about an average value is called the variation of dispersion of the data (Spiegel)
3. Dispersion or spread is the degree of the scatter or variation of the variable about a central value (Brooks & Dick)
4. The measurement of the scatterness of the mass of figures in a series about an average is called measure of variation or dispersion (Simpson & Kajka)

It is clear that dispersion (also known as scatter, spread or variation) measures the extent to which the items vary from some central value. Since measures of dispersion give an average of the differences of various items from an average, they are also called averages of the second order. An average is more meaningful when it is examined in the light of dispersion.

For example, if the average wage of the workers of factory A is Gh₵3885 and that of factory B Gh₵3900, we cannot necessarily conclude that the workers of factory B are better off because in factory B there may be much greater dispersion in the distribution of wages.

Significance of Measuring Variation
Measures of variation are needed for four basic purposes:
1. Measures of variation point out as to how far an average is representative of the mass. When dispersion is small, the average is a typical value in the sense that it closely represents the individual value and it is reliable in the sense that it is a good estimate of the average in the corresponding universe. On the other hand, when dispersion is large, the average is not so typical, and unless the sample is very large, the average may be quite unreliable.

2. Another purpose of measuring dispersion is to determine nature and cause of variation in order to control the variation itself. In matters of health variations in body temperature, pulse beat and blood pressure are the basic guides to diagnosis. Prescribed treatment is designed to control their variation. In industrial production efficient operation requires control of quality variation the causes of which are sought through inspection is basic to the control of causes of variation. In social sciences a special problem requiring the measurement of variability is the measurement of ‘inequality’ of the distribution of income or wealth.

3. Measures of dispersion enable a comparison to be made of two or more series with regard to their variability. The study of variation may also be looked upon as a means of determining uniformity of consistency. A high degree of variation would mean little uniformity or consistency whereas a low degree of variation would mean great uniformity or consistency.
4. Many powerful analytical tools in statistics such as correlation analysis, the testing of hypothesis, analysis of variance, the statistical quality control, regression analysis is based on measures of variation of one kind or another.

**Properties of a good measure of dispersion**

A good measure of dispersion should possess the following properties:

1. It should be simple to understand.
2. It should be easy to compute.
3. It should be rigidly defined.
4. It should be based on each and every item of the distribution.
5. It should be amenable to further algebraic treatment.
6. It should have sampling stability.
7. Extreme items should not unduly affect it.

**The Range**

The range is the simplest measure of variability to calculate, and one you have probably encountered many times in your life. The range is simply the highest score minus the lowest score.

The range is a measure of variability because it indicates the size of the interval over which the data points are distributed. A smaller range indicates less variability (less dispersion) among the data, whereas a larger range indicates the opposite.

The range of a data set is the number \( R \) defined by the formula, \( R = x_{\text{max}} - x_{\text{min}} \), where \( x_{\text{max}} \) is the largest measurement in the data set and \( x_{\text{min}} \) is the smallest.

Note that the range is the only measure of dispersion that has on average the reference point.

- When the data are not grouped the total amplitude is a difference between the largest and the smallest observed value: \( R = \text{Maximum X - X min} \).
- When data are grouped without class intervals still have \( R = \text{maximum X - X min} \).
- With class intervals the total amplitude is the difference between the upper boundary of the last class and the lower boundary of the first class. Then Range is \( \text{Lmax-Lmin} \).

The full range is inconvenient and only considers the two extreme values of the series, neglecting the set of intermediate values. We makes use of the full range when we want to determine the amplitude of the temperature in a day, quality control or as a quick calculation measure without much accuracy.

**Example**
1. What is the range of the following group of numbers: 10, 2, 5, 6, 7, 3, 4?

**Solution**
The highest number is 10, and the lowest number is 2
So 10 - 2 = 8
The range is 8.

2. Here is a dataset with 10 numbers: 99, 45, 23, 67, 45, 91, 82, 78, 62, 51. What is the range?

**Solution**
The highest number is 99 and the lowest number is 23, so 99 - 23 equals 76; the range is 76.

3. Find the range of each data set below:
Data Set I: 40, 38, 42, 40, 39, 39, 43, 40, 39, 40
Data Set II: 46, 37, 40, 33, 42, 36, 40, 47, 34, 45

**Solution**
For Data Set I the maximum is 43 and the minimum is 38, so the range is \( R = 43 - 38 = 5 \).
For Data Set II the maximum is 47 and the minimum is 33, so the range is \( R = 47 - 33 = 14 \).

4. Find the range for the values 40, 45, 48, 62, and 70.

**Solution**
\[ R = 70 - 40 = 30 \]

5. Find the range for the following three sets of data:
Set 1: 05, 15, 15, 05, 15, 05, 15, 15, 15, 15
Set 2: 8, 7, 15, 11, 12, 5, 13, 11, 15, 9
Set 3: 5, 5, 5, 5, 5, 5, 5, 5, 5, 5

**Solution**
In each of these three sets, the highest number is 15 and the lowest number is 5.
Since the range is the difference between the maximum value and the minimum value of the data, it is 10 in each case.
But the range fails to give any idea about the dispersal or spread of the series between the highest and the lowest value.
This becomes evident from the above data.

6. Use the table below to find the range of the values.
<table>
<thead>
<tr>
<th>Class</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>2</td>
<td>6</td>
<td>5</td>
<td>3</td>
</tr>
</tbody>
</table>

**Solution**

\[ R = 4 - 0 = 4 \]

7. Use the table below to find the range of the values.

<table>
<thead>
<tr>
<th>Interval</th>
<th>41-60</th>
<th>61-80</th>
<th>80-100</th>
</tr>
</thead>
<tbody>
<tr>
<td>Boundary</td>
<td>40.5-60.5</td>
<td>60.5-80.5</td>
<td>80.5-100.5</td>
</tr>
<tr>
<td>Frequency</td>
<td>6</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>

**Solution**

\[ R = 100.5 - 40.5 = 60 \]

8. Find the range for the following frequency distribution:

<table>
<thead>
<tr>
<th>Size</th>
<th>20-40</th>
<th>40-60</th>
<th>60-80</th>
<th>80-100</th>
<th>100-120</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>7</td>
<td>11</td>
<td>30</td>
<td>17</td>
<td>5</td>
</tr>
</tbody>
</table>

**Solution**

Here, the upper limit of the highest class is 120 and the lower limit of the lowest class is 20. Hence, the range is 120 - 20 = 100.

Note that the range is not influenced by the frequencies. Symbolically, the range is calculated by the formula \( L - S \), where \( L \) is the largest value and \( S \) is the smallest value in a distribution.

The coefficient of range is calculated by the formula: \( \frac{L - S}{L + S} \).

This is the relative measure.

9. Calculate the coefficient of range separately for the two sets of data given below:

Set 1: 8, 10, 20, 9, 15, 10, 13, 28

Set 2: 30, 35, 42, 50, 32, 49, 39, 33

**Solution**

It can be seen that the range in both the sets of data is the same:

Set 1: 28 - 8 = 20

Set 2: 50 - 30 = 20

Coefficient of range in Set 1 is: \( CR = \frac{28 - 8}{28 + 8} = 0.55 \)

Coefficient of range in set 2 is: \( CR = \frac{50 - 30}{50 + 30} = 0.25 \)

**Limitations of the range**
There are some limitations of range, which are as follows:
1. It is based only on two items and does not cover all the items in a distribution.
2. It is subject to wide fluctuations from sample to sample based on the same population.
3. It fails to give any idea about the pattern of distribution.
4. In the case of open-ended distributions, it is not possible to compute the range.

Despite these limitations of the range, it is mainly used in situations where one wants to quickly have some idea of the variability of a set of data.

When the sample size is very small, the range is considered quite a good measure of the variability. Thus, it is widely used in quality control where a continuous check on the variability of raw materials or finished products is needed.

The range is also a suitable measure in weather forecasting. The meteorological department uses the range by giving the maximum and the minimum temperatures. This information is quite useful to the common man, as he can know the extent of possible variation in the temperature on a particular day.

**The interquartile range or quartile deviation**
The interquartile range or the quartile deviation is a better measure of variation in a distribution than the range. Here, avoiding the 25 percent of the distribution at both the ends uses the middle 50 percent of the distribution. In other words, the interquartile range denotes the difference between the third quartile and the first quartile, expressed symbolically as $Q_3 - Q_1$.

Many times the interquartile range is reduced in the form of semi-interquartile range or quartile deviation, expressed as $(Q_3 - Q_1)/2$.

When quartile deviation is small, it means that there is a small deviation in the central 50 percent items. In contrast, if the quartile deviation is high, it shows that the central 50 percent items have a large variation. It may be noted that in a symmetrical distribution, the two quartiles, that is, $Q_3$ and $Q_1$ are equidistant from the median or $M - Q_1 = Q_3 - M$.

However, this is seldom the case as most of the education, business and economic data are asymmetrical. But, one can assume that approximately 50 percent of the observations are contained in the interquartile range. It may be noted that interquartile range or the quartile deviation is an absolute measure of dispersion. It can be changed into a relative measure of dispersion as $Q_3 - Q_1/Q_3 + Q_1$.

The computation of a quartile deviation is very simple, involving the computation of upper and lower quartiles. As the computation of the two quartiles has already been explained in the preceding chapter, it is not attempted here.
Merits of quartile deviation
The following merits are entertained by quartile deviation:
1. As compared to range, it is considered a superior measure of dispersion.
2. In the case of open-ended distribution, it is quite suitable.
3. Since it is not influenced by the extreme values in a distribution, it is particularly suitable in highly skewed or erratic distributions.

Limitations of quartile deviation
1. Like the range, it fails to cover all the items in a distribution.
2. It is not amenable to mathematical manipulation.
3. It varies widely from sample to sample based on the same population.
4. Since it is a positional average, it is not considered as a measure of dispersion. It merely shows a distance on scale and not a scatter around an average.

In view of the above-mentioned limitations, the interquartile range or the quartile deviation has a limited practical utility.

The Mean Deviation
The mean deviation is also known as the average deviation. As the name implies, it is the average of absolute amounts by which the individual items deviate from the mean.
Since the positive deviations from the mean are equal to the negative deviations, while computing the mean deviation, we ignore positive and negative signs.
Symbolically, $MD = \frac{\sum |d|}{n}$ for discrete data and $MD = \frac{\sum f |d|}{n}$ for grouped data, where MD is the mean deviation, f is the frequency, $|x|$ is the deviation of an item from the mean ignoring positive and negative signs, $n =$ the total number of observations.

Example

| Size  | Frequency | Midpoint x | fx  | d  | F|d| |
|-------|-----------|------------|-----|----|---|---|
| 2-4   | 20        | 3          | 60  | -2.6 |   | 52 |
| 4-6   | 40        |            |     |     |   |   |
| 6-8   | 30        |            |     |     |   |   |
| 8-10  | 10        |            |     |     |   |   |

Solution

$\bar{x} = \frac{\sum fx}{n} = \frac{560}{100} = 5.6$
<table>
<thead>
<tr>
<th>Class Interval</th>
<th>Frequency (f)</th>
<th>Midpoint (x)</th>
<th>Frequency Midpoint Product (fx)</th>
<th>Deviation (d)</th>
<th>Deviation Squared (d²)</th>
</tr>
</thead>
<tbody>
<tr>
<td>4-6</td>
<td>40</td>
<td>5</td>
<td>200</td>
<td>-0.6</td>
<td>0.36</td>
</tr>
<tr>
<td>6-8</td>
<td>30</td>
<td>7</td>
<td>210</td>
<td>1.4</td>
<td>1.96</td>
</tr>
<tr>
<td>8-10</td>
<td>10</td>
<td>9</td>
<td>90</td>
<td>-3.4</td>
<td>11.56</td>
</tr>
<tr>
<td>Total</td>
<td>100</td>
<td></td>
<td>560</td>
<td></td>
<td>15.2</td>
</tr>
</tbody>
</table>

\[ MD = \frac{\sum f |d|}{n} = \frac{152}{100} = 1.52 \]

**Merits of the mean deviation**
1. A major advantage of mean deviation is that it is simple to understand and easy to calculate.
2. It takes into consideration each and every item in the distribution. As a result, a change in the value of any item will have its effect on the magnitude of mean deviation.
3. The values of extreme items have less effect on the value of the mean deviation.
4. As deviations are taken from a central value, it is possible to have meaningful comparisons of the formation of different distributions.

**Limitations of the mean deviation**
1. It is not capable of further algebraic treatment.
2. At times it may fail to give accurate results. The mean deviation gives best results when deviations are taken from the median instead of from the mean.
But in a series, which has wide variations in the items, median is not a satisfactory measure.
3. Strictly on mathematical considerations, the method is wrong as it ignores the algebraic signs when the deviations are taken from the mean.

In view of these limitations, it is seldom used in business studies. A better measure known as the standard deviation is more frequently used.

**The Variance**
Variability can also be defined in terms of how close the scores in the distribution are to the middle of the distribution. Using the mean as the measure of the middle of the distribution, the variance is defined as the average squared difference of the scores from the mean.

This mean of the squared deviations is known as the variance. It may be noted that this variance is described by different terms that are used interchangeably, namely the variance of the distribution X the variance of X or the variance of the distribution.

To calculate the standard deviation, we need to calculate the variance first. The variance is the average of the squares of the deviations (the \( x - \bar{x} \) values for a sample, or the \( x - \mu \) values for a population).
The symbol $\sigma^2$ represents the population variance and the population standard deviation $\sigma$ is the square root of the population variance. The symbol $s^2$ represents the sample variance, and the sample standard deviation $s$ is the square root of the sample variance.

If the numbers come from a census of the entire population and not a sample, when we calculate the average of the squared deviations to find the variance, we divide by $N$, the number of items in the population.

If the data are from a sample rather than a population, when we calculate the average of the squared deviations, we divide by $n - 1$, one less than the number of items in the sample.

*Note:* Sample variance has different units from data. For example, if the units in a data set are inches, the new units would be inches squared, or square inches. It is thus primarily of theoretical importance and not considered for further statistical analysis.

**Variance of Discrete and Ungrouped Frequency Tables**

Formulas for the sample variance

$$s^2 = \frac{\sum (x - \bar{x})^2}{n - 1} \quad \text{or} \quad s^2 = \frac{\sum x^2 - (\sum x)^2}{n - 1}$$

$$s^2 = \frac{\sum f(x - \bar{x})^2}{n - 1}, \quad \text{or} \quad s^2 = \frac{\sum fx^2 - (\sum fx)^2}{n - 1}, \quad \text{where the denominator is} \ n - 1 \ \text{is the sample size minus 1.}$$

Although the first sets of formulas in each case look less complicated than the second, the latter are easier to use in hand computations, and are called a *shortcut formula*.

Formulas for the population variance

$$\sigma^2 = \frac{\sum (x - \mu)^2}{N} \quad \text{or} \quad \sigma^2 = \frac{\sum f(x - \mu)^2}{N}, \quad \text{where the denominator is} \ N \ \text{is the number of items in the population and} \ f \ \text{represents the frequency with which a value appears.}$$

Note that the denominator in the fraction is the full number of observations, not that number reduced by one, as is the case with the sample standard deviation. Since most data sets are samples, we will always work with the sample standard deviation and variance.

**Variance of Grouped Frequency Tables**
We determine the best estimate of the measures of centre by finding the mean of the grouped data with the formula \( \bar{x} = \frac{\sum fx}{\sum f} \) and the variance \( s^2 = \frac{\sum f(x-x)^2}{\sum f} \). or \( s^2 = \frac{\sum fx^2}{\sum f} - \left( \frac{\sum fx}{\sum f} \right)^2 \) where \( f \) is the interval frequencies and \( x \) is the interval midpoints.

Note that in many real-life situations the most important statistical issues have to do with comparing the means and standard deviations of data sets.

**Example**

1. Find the variance of the population data 20, 15, 19, 24, 16, 14.

**Solution**

\[
\mu = \frac{\sum x}{N} = \frac{108}{6} = 18
\]

<table>
<thead>
<tr>
<th>x</th>
<th>(x-(\mu))</th>
<th>(x-(\mu))^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>15</td>
<td>-3</td>
<td>9</td>
</tr>
<tr>
<td>19</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>24</td>
<td>6</td>
<td>36</td>
</tr>
<tr>
<td>16</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>14</td>
<td>-4</td>
<td>16</td>
</tr>
<tr>
<td><strong>108</strong></td>
<td></td>
<td><strong>70</strong></td>
</tr>
</tbody>
</table>

\[
\sigma^2 = \frac{\sum(x-\mu)^2}{N} = \frac{70}{6} = 11.6\bar{7}
\]

2. Assume the scores 1, 2, 4, and 5 were sampled from a larger population. Estimate the variance in the population.

**Solution**

\[
\bar{x} = \frac{\sum x}{n} = \frac{12}{4} = 3
\]

\[
s^2 = \frac{(x-\bar{x})^2}{n-1} = \frac{(1-3)^2 + (2-3)^2 + (4-3)^2 + (5-3)^2}{4-1}
\]

\[
s^2 = \frac{(1-3)^2 + (2-3)^2 + (4-3)^2 + (5-3)^2}{4-1} = \frac{4+1+1+4}{3} = \frac{10}{3} = 3.333
\]
3. Estimate the variance of 46, 37, 40, 33, 42, 36, 40, 47, 34, 45

Solution

\[
\bar{x} = \frac{\sum x}{n} = \frac{400}{10} = 40 \\
S^2 = \frac{\sum (x - \bar{x})^2}{n-1} = \frac{46 - 40)^2 + (37 - 40)^2 + (40 - 40)^2 + (36 - 40)^2 + \ldots + (45 - 40)^2}{10 - 1} \\
S^2 = \frac{6^2 + (-3)^2 + 0^2 + (-4)^2 + \ldots + 5^2}{10 - 1} \\
S^2 = \frac{36 + 9 + 0 + 36 + \ldots + 25}{9} = \frac{224}{9} = 24.8
\]

4. Use the short-cut formula to estimate the variance of 1.90, 3.00, 2.53, 3.71, 2.12, 1.76, 2.71, 1.39, 4.00, and 3.33.

Solution

\[
\sum x = 1.90 + 3.00 + 2.53 + 3.71 + 2.12 + 1.76 + 2.71 + 1.39 + 4.00 + 3.33 = 26.45 \\
\sum x^2 = 1.90^2 + 3.00^2 + 2.53^2 + 3.71^2 + 2.12^2 + 1.76^2 + 2.71^2 + 1.39^2 + 4.00^2 + 3.33^2 = 76.7321 \\
S^2 = \frac{\sum x^2 - (\sum x)^2}{n-1} = \frac{76.7321 - 26.45^2}{10 - 1} \\
S^2 = \frac{7.52427}{9} = 0.83603
\]

Although the variance is a measure of dispersion, the unit of its measurement is (points). If a distribution relates to income of families then the variance is (Gh₵)^2 and not cedis.

Similarly, if another distribution pertains to marks of students, then the unit of variance is (marks)^2.

To overcome this inadequacy, the square root of variance is taken, which yields a better measure of dispersion known as the standard deviation.

The standard deviation

It is the measure most commonly used dispersion because it takes into consideration all the values of the variable under study. It is an indicator of variability quite stable. The standard deviation is
based on deviations around the mean and its basic formula can be translated as the square root of the arithmetic mean of the squares of deviations.

- provides a numerical measure of the overall amount of variation in a data set
- can be used to determine whether a particular data value is close to or far from the mean.

**The standard deviation provides a measure of the overall variation in a data set**

The standard deviation is always positive or zero. The standard deviation is small when the data are all concentrated close to the mean, exhibiting little variation or spread. The standard deviation is larger when the data values are more spread out from the mean, exhibiting more variation.

Suppose that we are studying the amount of time customers wait in line at the checkout at supermarket A and supermarket B and the average wait time at both supermarkets is five minutes. At supermarket A, the standard deviation for the wait time is two minutes; at supermarket B the standard deviation for the wait time is four minutes.

Because supermarket B has a higher standard deviation, we know that there is more variation in the wait times at supermarket B. Overall, wait times at supermarket B are more spread out from the average; wait times at supermarket A are more concentrated near the average.

**The standard deviation can be used to determine whether a data value is close to or far from the mean**

Suppose that Rosa and Bintu both shop at supermarket A. Rosa waits at the checkout counter for seven minutes and Bintu waits for one minute.

At supermarket A, the mean waiting time is five minutes and the standard deviation is two minutes. The standard deviation can be used to determine whether a data value is close to or far from the mean.

**Uses of the Standard Deviation**

The standard deviation is similar to the mean deviation in that here too the deviations are measured from the mean. At the same time, the standard deviation is preferred to the mean deviation or the quartile deviation or the range because it has desirable mathematical properties.

The standard deviation is a frequently used measure of dispersion. It enables us to determine as to how far individual items in a distribution deviate from its mean.

In a symmetrical, bell-shaped curve:
(i) About 68 percent of the values will fall within: +1 standard deviation from the mean.
(ii) About 95 percent of the values will fall within +2 standard deviations from the mean.
(iii) About 99 percent of the values will fall within + 3 standard deviations from the mean.

Calculating the Standard Deviation
If \( x \) is a number, then the difference ‘\( x – \text{mean} \)’ is called its deviation. In a data set, there are as many deviations as there are items in the data set. The deviations are used to calculate the standard deviation.

If the numbers belong to a population, in symbols a deviation is \( x – \mu \). For sample data, in symbols a deviation is \( x – \bar{x} \).

The procedure to calculate the standard deviation depends on whether the numbers are the entire population or are data from a sample. The calculations are similar, but not identical. Therefore the symbol used to represent the standard deviation depends on whether it is calculated from a population or a sample.

The lower case letter ‘\( s \)’ represents the sample standard deviation and the Greek letter \( \sigma \) (sigma, lower case) represents the population standard deviation. If the sample has the same characteristics as the population, then ‘\( s \)’ should be a good estimate of \( \sigma \).

The standard deviation, \( s \) or \( \sigma \), is either zero or larger than zero. When the standard deviation is zero, there is no spread; that is, all the data values are equal to each other.

The standard deviation is small when the data are all concentrated close to the mean, and is larger when the data values show more variation from the mean.

When the standard deviation is a lot larger than zero, the data values are very spread out about the mean; outliers can make \( s \) or \( \sigma \) very large.

The standard deviation, when first presented, can seem unclear. By graphing your data, we can get a better ‘feel’ for the deviations and the standard deviation.

We will find that in symmetrical distributions, the standard deviation can be very helpful but in skewed distributions, the standard deviation may not be much help. The reason is that the two sides of a skewed distribution have different spreads.
In a skewed distribution, it is better to look at the first quartile, the median, the third quartile, the smallest value, and the largest value. Because numbers can be confusing, always graph your data in a histogram or a box plot.

**Standard deviation of Discrete and Ungrouped Frequency Tables**

**Formulas for the Sample Standard Deviation**

\[ s = \sqrt{\frac{\sum (x - \overline{x})^2}{n - 1}} \quad \text{or} \quad s = \sqrt{\frac{\sum f(x - \overline{x})^2}{n - 1}}, \]  

where the denominator is \( n - 1 \) is the sample size MINUS 1.

**Formulas for the Population Standard Deviation**

\[ \sigma = \sqrt{\frac{\sum (x - \mu)^2}{N}} \quad \text{or} \quad \sigma = \sqrt{\frac{\sum f(x - \mu)^2}{N}}, \]  

where the denominator is \( N \) is the number of items in the population and \( f \) represents the frequency with which a value appears.

**Example**

1. In a class five, the teacher was interested in the average age and the sample standard deviation of the ages of her students. The following data are the ages for a SAMPLE of 20 class five pupils. The ages are rounded to the nearest half year: 9; 9.5; 9.5; 10; 10; 10; 10; 10; 10.5; 10.5; 10.5; 10.5; 11; 11; 11; 11; 11; 11.5; 11.5; 11.5.

   Calculate the average age and the sample standard deviation of the ages of her pupils.

   **Solution**

\[
\overline{x} = \frac{\sum x}{n} = \frac{9 + 9.5(2) + 10(4) + 10.5(4) + 11(6) + 11.5(4)}{20} = 10.525
\]

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f )</th>
<th>( x )-mean</th>
<th>((x\text{-mean})^2)</th>
<th>( f(x\text{-mean})^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>9</td>
<td>1</td>
<td>9 – 10.525 = -1.525</td>
<td>((-1.525)^2 = 2.325625)</td>
<td>(1 \times 2.325625 = 2.325625)</td>
</tr>
<tr>
<td>9.5</td>
<td>2</td>
<td>9.5 – 10.525 = -1.025</td>
<td>((-1.025)^2 = 1.050625)</td>
<td>(2 \times 1.050625 = 2.101250)</td>
</tr>
<tr>
<td>10</td>
<td>4</td>
<td>10 – 10.525 = -0.525</td>
<td>((-0.525)^2 = 0.275625)</td>
<td>(4 \times 0.275625 = 1.1025)</td>
</tr>
<tr>
<td>10.5</td>
<td>4</td>
<td>10.5 – 10.525 = -0.025</td>
<td>((-0.025)^2 = 0.000625)</td>
<td>(4 \times 0.000625 = 0.0025)</td>
</tr>
<tr>
<td>11</td>
<td>6</td>
<td>11 – 10.525 = 0.475</td>
<td>((0.475)^2 = 0.225625)</td>
<td>(6 \times 0.225625 = 1.35375)</td>
</tr>
<tr>
<td>11.5</td>
<td>3</td>
<td>11.5 – 10.525 = 0.975</td>
<td>((0.975)^2 = 0.950625)</td>
<td>(3 \times 0.950625 = 2.851875)</td>
</tr>
<tr>
<td>Total</td>
<td>20</td>
<td>(11.5 – 10.525 = 0.975)</td>
<td>(3 \times 0.950625 = 2.851875)</td>
<td></td>
</tr>
</tbody>
</table>

\[
s = \sqrt{\frac{\sum (x - \overline{x})^2}{n - 1}} = \frac{9.7375}{20 - 1} = 0.5125
\]

2. Find the standard deviation of the population data 20, 15, 19, 24, 16, 14.

   **Solution**
\[ \mu = \frac{\sum x}{N} = \frac{108}{6} = 18 \]

<table>
<thead>
<tr>
<th>(x)</th>
<th>((x-\mu))</th>
<th>((x-\mu)^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>15</td>
<td>-3</td>
<td>9</td>
</tr>
<tr>
<td>19</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>24</td>
<td>6</td>
<td>36</td>
</tr>
<tr>
<td>16</td>
<td>-2</td>
<td>4</td>
</tr>
<tr>
<td>14</td>
<td>-4</td>
<td>16</td>
</tr>
<tr>
<td><strong>108</strong></td>
<td></td>
<td><strong>70</strong></td>
</tr>
</tbody>
</table>

\[ \sigma^2 = \frac{\sum(x-\mu)^2}{N} = \frac{70}{6} = 11.67 \]

\[ \sigma = \sqrt{\sigma^2} = \sqrt{\left(\frac{\sum(x-\mu)^2}{N}\right)} = \sqrt{\frac{70}{6}} = \sqrt{11.67} = 3.42 \]

**Explanation of the standard deviation calculation shown in the table**

The deviations show how spread out the data are about the mean. The data value 11.5 is farther from the mean than is the data value 11 which is indicated by the deviations 0.97 and 0.47.

A positive deviation occurs when the data value is greater than the mean, whereas a negative deviation occurs when the data value is less than the mean. The deviation is \(-1.525\) for the data value nine.

If you add the deviations, the sum is always zero. So you cannot simply add the deviations to get the spread of the data. By squaring the deviations, you make them positive numbers, and the sum will also be positive. The variance, then, is the average squared deviation.

The variance is a squared measure and does not have the same units as the data. Taking the square root solves the problem. The standard deviation measures the spread in the same units as the data.

Notice that instead of dividing by \(n = 20\), the calculation divided by \(n - 1 = 20 - 1 = 19\) because the data is a sample. For the sample variance, we divide by the sample size minus one \((n - 1)\). Why not divide by \(n\)? The answer has to do with the population variance. The sample variance is an estimate of the population variance. Based on the theoretical mathematics that lies behind these calculations, dividing by \((n - 1)\) gives a better estimate of the population variance.

**Standard deviation of Grouped Frequency Tables**

Recall that for grouped data we do not know individual data values, so we cannot describe the typical value of the data with precision. In other words, we cannot find the exact mean, median, or
mode. We can, however, determine the best estimate of the measures of centre by finding the mean of the grouped data with the formula \( \bar{x} = \frac{\sum fx}{\sum f} \) and the standard deviation 

\[ s = \sqrt{\frac{\sum f x^2}{\sum f} - \left( \frac{\sum fx}{\sum f} \right)^2} \] or 

\[ s = \sqrt{\frac{\sum f x^2}{\sum f} - \bar{x}^2} \] where \( f \) is the interval frequencies and \( x \) is the interval midpoints.

When it becomes clear that the actual mean would turn out to be in fraction, calculating deviations from the mean would be too cumbersome. In such cases, an assumed mean is used and the deviations from it are calculated. In doing so, while midpoint of any class can be taken as an assumed mean, it is advisable to choose the mid-point of that class that would make calculations least cumbersome.

Also, just as we could not find the exact mean, neither can we find the exact standard deviation. Remember that standard deviation describes numerically the expected deviation a data value has from the mean. In simple English, the standard deviation allows us to compare how “unusual” individual data is compared to the mean.

**Example**

1. Find the standard deviation for the data in **Table below**

<table>
<thead>
<tr>
<th>Class</th>
<th>0-2</th>
<th>3-5</th>
<th>6-8</th>
<th>9-11</th>
<th>12-14</th>
<th>15-17</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency</td>
<td>1</td>
<td>6</td>
<td>10</td>
<td>7</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

**Solution**

<table>
<thead>
<tr>
<th>Class</th>
<th>( f )</th>
<th>( x )</th>
<th>( x^2 )</th>
<th>( fx )</th>
<th>( f x^2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>3-5</td>
<td>6</td>
<td>4</td>
<td>16</td>
<td>24</td>
<td>96</td>
</tr>
<tr>
<td>6-8</td>
<td>10</td>
<td>7</td>
<td>49</td>
<td>70</td>
<td>490</td>
</tr>
<tr>
<td>9-11</td>
<td>7</td>
<td>10</td>
<td>100</td>
<td>70</td>
<td>700</td>
</tr>
<tr>
<td>12-14</td>
<td>0</td>
<td>13</td>
<td>169</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>15-17</td>
<td>2</td>
<td>16</td>
<td>256</td>
<td>32</td>
<td>512</td>
</tr>
<tr>
<td>Total</td>
<td>26</td>
<td>16</td>
<td>256</td>
<td>197</td>
<td>1,799</td>
</tr>
</tbody>
</table>

\[ \bar{x} = \frac{\sum fx}{\sum f} = \frac{197}{26} = 7.577 \]
\[ s = \sqrt{\frac{\sum fx^2}{\sum f} - \left(\frac{\sum fx}{\sum f}\right)^2} = \sqrt{\frac{1,799}{26} - \left(\frac{197}{26}\right)^2} \]

\[ s = \sqrt{\frac{1,799}{26} - \left(\frac{197}{26}\right)^2} = \sqrt{69.192 - (7.577)^2} \]

\[ s = \sqrt{69.192 - (7.577)^2} = \sqrt{69.192 - 42.257} = \sqrt{26.935} = 3.5 \]

For this data set, we have the mean, \( \bar{x} = 7.577 \) and the standard deviation, \( s = 3.5 \). This means that a randomly selected data value would be expected to be 3.5 units from the mean. If we look at the first class, we see that the class midpoint is equal to one. This is almost two full standard deviations from the mean since \( 7.58 - 3.5 - 3.5 = 0.58 \).

2. The following distribution relates to marks obtained by students in an examination. Use it to find the standard deviation.

<table>
<thead>
<tr>
<th>Marks</th>
<th>Frequency f</th>
<th>Midpoints x</th>
<th>( d = \text{Deviation}/c = d/10 )</th>
<th>fd</th>
<th>fd^2</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-10</td>
<td>1</td>
<td>5</td>
<td>-5</td>
<td>-5</td>
<td>25</td>
</tr>
<tr>
<td>10-20</td>
<td>3</td>
<td>15</td>
<td>-4</td>
<td>-12</td>
<td>48</td>
</tr>
<tr>
<td>20-30</td>
<td>6</td>
<td>25</td>
<td>-3</td>
<td>-18</td>
<td>54</td>
</tr>
<tr>
<td>30-40</td>
<td>10</td>
<td>35</td>
<td>-2</td>
<td>-20</td>
<td>40</td>
</tr>
<tr>
<td>40-50</td>
<td>12</td>
<td>45</td>
<td>-1</td>
<td>-12</td>
<td>12</td>
</tr>
<tr>
<td>50-60</td>
<td>11</td>
<td>55</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>60-70</td>
<td>6</td>
<td>65</td>
<td>1</td>
<td>6</td>
<td>6</td>
</tr>
<tr>
<td>70-80</td>
<td>3</td>
<td>75</td>
<td>2</td>
<td>6</td>
<td>12</td>
</tr>
<tr>
<td>80-90</td>
<td>2</td>
<td>85</td>
<td>3</td>
<td>6</td>
<td>18</td>
</tr>
<tr>
<td>90-100</td>
<td>1</td>
<td>95</td>
<td>4</td>
<td>4</td>
<td>16</td>
</tr>
<tr>
<td>Total</td>
<td>55</td>
<td></td>
<td></td>
<td>-45</td>
<td>231</td>
</tr>
</tbody>
</table>

\[ \sigma = \sqrt{\frac{\sum fd^2}{N} - \left(\frac{\sum fd}{N}\right)^2} = \sqrt{\frac{231}{55} - \frac{-45}{55}} = 18.8 \]

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The standard deviation is an absolute measure of dispersion as it measures variation in the same units as the original data. As such, it cannot be a suitable measure while comparing two or more distributions.

For this purpose, we should use a relative measure of dispersion. One such measure of relative dispersion is the coefficient of variation, which relates the standard deviation and the mean such that the standard deviation is expressed as a percentage of mean. Thus, the specific unit in which the standard deviation is measured is done away with and the new unit becomes percent.

**Coefficient of variation and Pearson’s Coefficient of Variation**

The Pearson’s coefficient of variation is given by \( CV = \frac{s}{\mu} \times 100 \).

If \( CV < 15\% \), we have a low dispersion

If \( 15\% < CV < 30\% \), we have an average dispersion

If \( CV > 30\% \), we have a high dispersion

**Example**

In a small business firm, two typists are employed—typist A and typist B. Typist A types out, on an average, 30 pages per day with a standard deviation of 6. Typist B, on an average, types out 45 pages with a standard deviation of 10. Which typist shows greater consistency in his output?

**Solution**

Coefficient of variation for \( A = \frac{\sigma}{\mu} \times 100 = \frac{6}{30} \times 100 = 20\% \)

Coefficient of variation for \( B = \frac{\sigma}{\mu} \times 100 = \frac{10}{45} \times 100 = 22.2\% \)

**Comment**

These calculations clearly indicate that although typist B types out more pages, there is a greater variation in his output as compared to that of typist A.

Alternatively, we can say though typist A’s daily output is much less, he is more consistent than typist B.

This usefulness of the coefficient of variation becomes clear in comparing two groups of data having different means.

**Standardised Variable, Standard Scores (Z-scores)**
The empirical rule tells us that for a bell-shaped distribution, it is unusual for an observation to fall more than 3 standard deviations from the mean. An alternative criterion for identifying potential outliers uses the standard deviation. An observation in a bell-shaped distribution is regarded as a potential outlier if it falls more than 3 standard deviations from the mean.

The \( z \)-score allows us to quickly tell how surprising or extreme an observation is. The \( z \)-score converts an observation (regardless of the observation’s unit of measurement) to a common scale of measurement, which allows comparisons.

The **\( z \)-score** for an observation is the number of standard deviations that it falls from the mean. A positive \( z \)-score indicates the observation is above the mean. A negative \( z \)-score indicates the observation is below the mean.

Another way to locate a particular observation \( x \) in a data set is to compute its distance from the mean in units of standard deviation. The **\( z \)-score of an observation** \( x \) **is the number** \( z \) **given by the computational formula**

\[
    z = \frac{x - \bar{x}}{s} \quad \text{or} \quad z = \frac{X - \mu}{\sigma},
\]

according to whether the data set is a sample or is the entire population.

The formulas in the definition allow us to compute the \( z \)-score when \( x \) is known. If the \( z \)-score is known then \( x \) can be recovered using the corresponding inverse formulas

\[
    x = \bar{x} - sz \quad \text{or} \quad X = \mu + \sigma z.
\]

The \( z \)-score indicates how many standard deviations an individual observation \( x \) is from the centre of the data set, its mean. If \( z \) is negative then \( x \) is below average. If \( z \) is 0 then \( x \) is equal to the average. If \( z \) is positive then \( x \) is above average.
The variable \( Z = (x - \mu)/\sigma \), which measures the deviation from the mean in units of the standard deviation, is called a standardised variable. Since both the numerator and the denominator are in the same units, a standardised variable is independent of units used.

If deviations from the mean are given in units of the standard deviation, they are said to be expressed in standard units or standard scores. Through this concept of standardised variable, proper comparisons can be made between individual observations belonging to two different distributions whose compositions differ.

The standard deviation is useful when comparing data values that come from different data sets. If the data sets have different means and standard deviations, then comparing the data values directly can be misleading. For each data value, calculate how many standard deviations away from its mean the value is.

**Example**
1. Find the \( z \)-scores for all ten observations in the GPA sample data 1.90 3.00 2.53 3.71 2.12 1.76 2.71 1.39 4.00 3.33.

**Solution**
For these data \( \bar{x} = 2.645 \) and \( s = 8.8674 \). The first observation \( x = 1.9 \) in the data set has \( z \)-score
\[
Z = \frac{x - \bar{x}}{s} = \frac{1.9 - 2.645}{0.8674} = -0.8589,
\]
which means that \( x = 1.90 \) is 0.8589 standard deviations below the sample mean.

The second observation \( x = 3.00 \) has \( z \)-score:
\[
Z = \frac{x - \bar{x}}{s} = \frac{3.00 - 2.645}{0.8674} = 0.4093,
\]
which means that \( x = 3.00 \) is 0.4093 standard deviations above the sample mean.

Repeating the process for the remaining observations gives the full set of \( z \)-scores \(-0.86; 0.41; -0.13; 1.23; -0.61; -1.02; 0.07; -1.45; 1.56; 0.79\).

2. Suppose the mean and standard deviation of the GPAs of all currently registered students at a college are \( \mu = 2.74 \) and \( \sigma = 0.50 \).

The \( z \)-scores of the GPAs of two students, Antonio and Beatrice, are \( z = -0.62 \) and \( z = 1.28 \), respectively. What are their GPAs?

**Solution**
Using the formula \( z = \frac{x - \mu}{\sigma} \) or \( X = \mu + \sigma z \) of z-scores we compute the GPAs as follows:

**Antonion:** 
\[
x = \mu + \sigma z = 2.70 + (-0.62)(0.50) = 2.39
\]

**Beatrice:** 
\[
x = \mu + \sigma z = 2.70 + (1.28)(0.50) = 3.34
\]

3. A student has scored 68 marks in Statistics for which the average marks were 60 and the standard deviation was 10. In the paper on Mathematics, she scored 74 marks for which the average marks were 68 and the standard deviation was 15. In which paper, Statistics or Mathematics, was she relative standing higher?

**Solution**
The standardised variable \( Z = \frac{(x - \mu)}{\sigma} \) measures the deviation of \( x \) from the mean \( \mu \) in terms of standard deviation \( \sigma \).

For Statistics, 
\[
Z = \frac{(68 - 60)}{10} = 0.8
\]

For Mathematics, 
\[
Z = \frac{(74 - 68)}{15} = 0.4
\]

Since the standard score is 0.8 in Statistics as compared to 0.4 in Mathematics, her relative standing was higher in Statistics.

4. Convert the set of numbers 6, 7, 5, 10 and 12 into standard scores.

**Solution**
\[
\bar{x} = \frac{\sum x}{n} = \frac{40}{5} = 8 \quad \text{and} \quad \sigma = \frac{\sum (x - \bar{x})^2}{n} = 2.61
\]

\[
z_6 = \frac{x - \bar{x}}{\sigma} = \frac{6 - 8}{2.61} = -0.77 \quad z_7 = \frac{x - \bar{x}}{\sigma} = \frac{7 - 8}{2.61} = -0.38
\]

\[
z_5 = \frac{x - \bar{x}}{\sigma} = \frac{5 - 8}{2.61} = -1.15 \quad z_{10} = \frac{x - \bar{x}}{\sigma} = \frac{6 - 8}{2.61} = 0.77 \quad z_{12} = \frac{x - \bar{x}}{\sigma} = \frac{12 - 8}{2.61} = 1.53
\]

Thus the standard scores for 6, 7, 5, 10 and 12 are -0.77, -0.38, -1.15, 0.77 and 1.53, respectively.

5. Let’s consider air pollution data for the European Union (EU). The Energy-EU data file 8 on the text CD contains data on per capita carbon dioxide (CO₂) emissions, in metric tons, for the 27 nations in the EU. The mean was 8.3 and the standard deviation was 3.6.

a. How many standard deviations from the mean was the CO₂ value of 21.3 for Luxembourg?

b. The CO₂ value for the United States was 18.9. According to the three-standard-deviation criterion, is the United States an outlier on carbon dioxide emissions relative to the EU?

**Solution**
a). Since the mean is 8.3 and \( s = 3.6 \) inches, the \( z \)-score for the observation of
21.3 is \[ z = \frac{x - \bar{x}}{s} = \frac{21.3 - 8.3}{3.6} = 3.6 \]

The carbon dioxide emission (per capita) for Luxembourg is 3.6 standard deviations above the mean. By the 3 standard deviation criterion, this is a potential outlier. Since it is well removed from the rest of the data, we would regard it as an actual outlier. However, Luxembourg has only 350,000 people, so in terms of the amount of pollution it is not a major polluter in the EU.

b). The \( z \)-score for the CO\(_2\) value of the United States is \[ z = \frac{x - \bar{x}}{s} = \frac{18.9 - 8.3}{3.6} = 2.9 \]

Although the 3 standard deviation rule fails to flag the United States as an outlier relative to EU nations, the value of 2.9 is close enough to 3 to garner some attention. Furthermore, because of the relatively large size of the U.S. population, a \( z \)-score this close to 3 indicates that the U.S. is a significant contributor to overall CO\(_2\) emission.

**Insight**

The \( z \)-scores of 3.6 and 2.9 are positive. This indicates that the observations are above the mean, because an observation above the mean has a positive \( z \)-score. In fact, these large positive \( z \)-scores tell us that Luxembourg and the United States have very high CO\(_2\) emissions compared to the other nations. The \( z \)-score is negative when the observation is below the mean. For instance, France has a CO\(_2\) value of 5.7, which is below the mean of 8.3 and has a \( z \)-score of -0.7.

**Example**

Two students, John and Ali, from different high schools, wanted to find out who had the highest GPA when compared to their schools. Which student had the highest GPA when compared to his school?

<table>
<thead>
<tr>
<th>Student</th>
<th>School GPA</th>
<th>Mean</th>
<th>School SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>John</td>
<td>2.85</td>
<td>3.00</td>
<td>0.7</td>
</tr>
<tr>
<td>Ali</td>
<td>77</td>
<td>80</td>
<td>10</td>
</tr>
</tbody>
</table>

**Solution**

For each student, determine how many standard deviations of his GPA is away from the average, for his school. Pay careful attention to signs when comparing and interpreting the answer.

\[ s_{John} = \frac{x - \mu}{\sigma} = \frac{2.85 - 3.00}{0.7} = -0.21 \]

\[ s_{Ali} = \frac{x - \mu}{\sigma} = \frac{77 - 80}{10} = -0.30 \]

John has the better GPA when compared to his school because his GPA is 0.21 standard deviations below his school’s mean while Ali’s GPA is 0.3 standard deviations below his school’s mean.
John's $z$-score of $-0.21$ is higher than Ali's $z$-score of $-0.3$. For GPA, higher values are better, so we conclude that John has the better GPA when compared to his school.

Activities 2
1. Find the $z$-score of each measurement in the following sample data set $-5, 6, 2, -1, 0$.
   Answer
   $-1.3, 1.39, 0.4, -0.35, -0.11$
2. For the population $0, 0, 2, 2$, compute each of the following:
   a. The population mean $\mu$.
   b. The population variance $\sigma^2$.
   c. The population standard deviation $\sigma$.
   d. The $z$-score for every value in the population data set.
   Answer
   a. 1.
   b. 1.
   c. 1.
   d. $z = -1$ for $x = 0$, $z = 1$ for $x = 2$.
3. A measurement $x$ in a sample with mean $\bar{x} = 10$ and standard deviation $s = 3$ has $z$-score $z = 2$. Find $x$.
   Answer
   $x = 16$
4. A measurement $x$ in a population with mean $\mu = 2.3$ and standard deviation $\sigma = 1.3$ has $z$-score $z = 2$. Find $x$.
   Answer
   $x = 4.9$
5. The weekly sales for the last 20 weeks in a kitchen appliance store for an electric automatic rice cooker are $20, 15, 15, 19, 15, 19, 17, 15, 14, 12, 16, 15, 14, 13, 16, 16, 18, 9, 18, 15$.
   a. Find the percentile rank of 15.
   b. If the sample accurately reflects the population, then what percentage of weeks would an inventory of 15 rice cookers be adequate?
   Answer
   a. 55.
   b. 55.
Activities
1. Find the range, the variance, and the standard deviation for the sample of ten IQ scores randomly selected from a school for academically gifted students: 132, 139, 162, 147, 133, 160, 145, 150, 148 and 153

Answers
\[ R = 30, \quad s^2 = 103.2, \quad s = 10.2. \]

The interquartile range
Another name for fourths is hinges. The difference between the upper and lower fourths (called the fourth-spread) should be close but not necessarily equal to the interquartile range since the quartiles are not necessarily equal to the fourths. Quartiles or fourths are often used when the distribution is skewed or outliers are expected.

The interquartile range is a number that indicates the spread of the middle half or the middle 50% of the data. It is the difference between the third quartile (Q3) and the first quartile (Q1).
\[ \text{IQR} = Q_3 - Q_1. \]

The interquartile range (IQR) is the quantity \( \text{IQR} = Q_3 - Q_1. \)

The Deviation quartile
Also called semi-interquartile range and is based on quartiles.
\[ SQR = \frac{Q_3 - Q_1}{2} \]

Remarks:
1. The quartile deviation has the advantage the fact that it is an easy measure to calculate and interpret. Besides, is not affected by extreme, large or small values and is recommended, therefore, when between the data contained extreme values are not considered representative.

2. The quartile deviation should be used preferably when the measure of central tendency is the median.

3. It is a measure insensitive to the distribution of smaller items that Q1, between Q1 and Q3 and higher than Q3.

Example
For the values 40, 45, 48, 62 and 70, find the quartile deviation.

Solution
\[ Q_1 = (45 + 40) / 2 = 42.5 \text{ and } Q_3 = (70 + 62) / 2 = 66 \]
\[ SQR = (66 \text{ to } 42.5) / 2 = 11.75 \]

The standard error of the mean
How much the statistic varies from one sample to another is known as the sampling variability of a statistic. We typically measure the sampling variability of a statistic by its standard error.

The *standard error of the mean* or the standard error is a special standard deviation and is known as the standard deviation of the sampling distribution of the mean.

The notation for the standard error of the mean is \( s_\sigma = \frac{\sigma}{\sqrt{n}} \), where \( \sigma \) is the standard deviation of the population and \( n \) is the size of the sample.

**Lorenz Curve**

This measure of dispersion is graphical. It is known as the Lorenz curve named after Dr. Max Lorenz. It is generally used to show the extent of concentration of income and wealth.

The Lorenz curve is a simple graphical device to show the disparities of distribution in any phenomenon. It is, used in business and economics to represent inequalities in income, wealth, production, savings, and so on.
The straight line AB is a line of equal distribution, whereas AEB shows complete inequality. Curve ACB and curve ADB are the Lorenz curves. As curve ACB is nearer to the line of equal distribution, it has more equitable distribution of income than curve ADB. Assuming that these two curves are for the same company, this may be interpreted in a different manner. Prior to taxation, the curve ADB showed greater inequality in the income of its employees. After the taxation, the company’s data resulted into ACB curve, which is closer to the line of equal distribution. In other words, as a result of taxation, the inequality has reduced.

The steps involved in plotting the Lorenz curve are as follows:
1. Convert a frequency distribution into a cumulative frequency table.
2. Calculate percentage for each item taking the total equal to 100.
3. Choose a suitable scale and plot the cumulative percentages of the persons and income. Use the horizontal axis of X to depict percentages of persons and the vertical axis of Y to depict percentages of income.
4. Show the line of equal distribution, which will join 0 of X-axis with 100 of Y-axis.
5. The curve obtained in (3) above can now be compared with the straight line of equal distribution obtained in (4) above.

If the Lorenz curve is close to the line of equal distribution, then it implies that the dispersion is much less. If, on the contrary, the Lorenz curve is farther away from the line of equal distribution, it implies that the dispersion is considerably wide.

**Graphical Descriptive Statistics**
Statistics is a mathematical science that is concerned with the collection, analysis, interpretation or explanation, and presentation of data. Insights from data may come from a well conceived visualization of the data, from modern methods of statistical learning and model selection as well as from time-honoured formal statistical procedures.

The first encounters one has to data are through graphical displays and numerical summaries. A well-known adage is that “a picture is worth a thousand words.” This saying proves true when it comes to presenting statistical information in a data set. There are many effective ways to present data graphically. The graphical tools are among the most commonly used and are relevant to the subsequent presentations of statistics.

The goal is to find an elegant method for this presentation that is at the same time both objective and informative—making clear with a few lines or a few numbers the salient features of the data. In this sense, data presentation is at the same time an art, a science, and an obligation to impartiality. In the section, we will describe some of the standard graphical presentations of data.

**Graphs for categorical data**
Pie Chart
A pie chart is a circular chart divided into sectors, illustrating relative magnitudes in frequencies or percents. In a pie chart, the area is proportional to the quantity it represents.

Bar Chart
Because the human eye is good at judging linear measures and poor at judging relative areas, a bar chart or bar graph is often preferable to pie charts as a way to display categorical data.

Two-way Tables
Relationships between two categorical variables can be shown through a two-way table (also known as a contingency table, cross tabulation table or a cross classifying table).

Examples
1. In 1964, a Surgeon, General Dr. Luther Leonidas Terry published a landmark report saying that smoking may be hazardous to health. This led to many influential reports on the topic, including the study of the smoking habits of 5,375 high school children in Tucson in 1967. Here is a two-way table summarizing some of the results.

<table>
<thead>
<tr>
<th></th>
<th>Student smokes</th>
<th>Student does not smoke</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>2 parents smoke</td>
<td>400</td>
<td>1380</td>
<td>1780</td>
</tr>
<tr>
<td>1 parent smokes</td>
<td>416</td>
<td>1823</td>
<td>2239</td>
</tr>
<tr>
<td>0 parents smoke</td>
<td>188</td>
<td>1168</td>
<td>1356</td>
</tr>
<tr>
<td>Total</td>
<td>4004</td>
<td>4371</td>
<td>5375</td>
</tr>
</tbody>
</table>

- The row variable is the parents smoking habits.
- The column variable is the student smoking habits.
- The cells display the counts for each of the categories of row and column variables.

A two-way table with r rows and c columns is often called an r by c table (written r x c). The totals along each of the rows and columns give the marginal distributions.

2. Hemoglobin E is a variant of hemoglobin with a mutation in the gene causing substitution of glutamic acid. The glutamic acid is the second most common abnormal hemoglobin after sickle cell hemoglobin (HbS). The table below gives the hemoglobin genotypes on two Ghanaians.

<table>
<thead>
<tr>
<th>Genotype</th>
<th>AA</th>
<th>AE</th>
<th>EE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Flores</td>
<td>128</td>
<td>6</td>
<td>0</td>
</tr>
<tr>
<td>Sumba</td>
<td>116</td>
<td>78</td>
<td>4</td>
</tr>
</tbody>
</table>

Because the heterozygotes are rare on Flores, it appears malaria is less prevalent there since the heterozygote does not provide an adaptive advantage.

Stem-and-leaf diagrams
One way is to construct a *stem and leaf* diagram. The numbers in the tens place, start from 2 through 9, are the “stems,” and are arranged in numerical order from top to bottom to the left of a vertical line.

The number in the units place in each measurement is a “leaf,” and is placed in a row to the right of the corresponding stem. Thus the three leaves 5, 8 and 9 in the row headed with the stem 6 correspond to the three exam scores in the 60s, 65 (in the first row of data), 68 (in the third row), and 69 (also in the third row).

The display is made even more useful for some purposes by rearranging the leaves in numerical order. Either way, with the data reorganized certain information of interest becomes apparent immediately.

There are two perfect scores; three students made scores under 60; most students scored in the 70s, 80s and 90s; and the overall average is probably in the high 70s or low 80s.

**Example**

Suppose 30 students in a statistics class took a test and made the following scores: 86, 90, 40, 80, 83, 58, 25, 70, 68, 77, 73, 69, 73, 73, 100, 76, 70, 78, 100, 90, 87, 90, 83, 97, 69, 71, 92, 93, 95, 74.

**Solutions**

How did the class do on the test?

A quick glance at the set of 30 numbers does not immediately give a clear answer. However, the data set may be reorganized and rewritten to make relevant information more visible.

```
  2 | 5
  3 |
  4 | 0
  5 | 8
  6 | 9 8 9
  7 | 7 3 6 0 3 3 0 1 8 4
  8 | 6 0 3 3 7
  9 | 0 3 0 0 5 7 2
 10 | 0 0
```

In this example the scores have a natural stem (the tens place) and leaf (the ones place). One could spread the diagram out by splitting each tens place number into lower and upper categories. For example, all the scores in the 80s may be represented on two separate stems, lower 80s and upper 80s:
The definitions of stems and leaves are flexible in practice. The general purpose of a stem and leaf diagram is to provide a quick display of how the data are distributed across the range of their values; some improvisation could be necessary to obtain a diagram that best meets that goal.

Note that all of the original data can be recovered from the stem and leaf diagram. This will not be true of other types of graphical displays.

**Frequency histograms**

The stem and leaf diagram is not practical for large data sets, so we need a different, purely graphical way to represent data.

Histograms are a common visual representation of a quantitative variable. Histograms summarize the data using rectangles to display either frequencies or proportions as normalized frequencies.

In making a histogram, we:

- Divide the range of data into bins of equal width (usually, but not always).
- Count the number of observations in each class.
- Draw the histogram rectangles representing frequencies or percents by area.

Interpret the histogram by giving:

- the overall pattern
- the centre
- the spread
- the shape (symmetry, skewness, peaks)
- deviations from the pattern
- outliers
- gaps

The direction of the skewness is the direction of the longer of the two tails (left or right) of the distribution.

No one choice for the number of bins is considered best. One possible choice for larger data sets is Sturges’ formula to choose \( 1 + \log_2 n \) bins.

The floor function is obtained by rounding down to the next integer.

**Example**
Suppose 30 students in a statistics class took a test and made the following scores: 86, 90, 40, 80, 83, 58, 70, 68, 77, 73, 69, 73, 73, 100, 76, 70, 78, 100, 90, 87, 90, 83, 97, 69, 71, 92, 93, 95, 74.

**Solutions**
For the 30 scores on the exam, it is natural to group the scores on the standard ten-point scale, and count the number of scores in each group. Thus there are two 100s, seven scores in the 90s, six in the 80s, and so on.

We then construct the *frequency histogram* by drawing for each group, or class, a vertical bar whose length is the number of observations in that group.

For example, the bar labelled 100 is 2 units long, the bar labelled 90 is 7 units long, and so on.

While the individual data values are lost, we know the number in each class. This number is called the *frequency* of the class, hence the name frequency histogram as follows:

![Frequency Histogram](image)

The same procedure can be applied to any collection of numerical data. Observations are grouped into several classes and the frequency (the number of observations) of each class is noted.

These classes are arranged and indicated in order on the horizontal axis (called the *x*-axis), and for each group a vertical bar, whose length is the number of observations in that group, is drawn.
The resulting display is a frequency histogram for the data. Frequency Histogram is apparent, particularly if you imagine turning the stem and leaf diagram on its side by rotating it a quarter turn counter-clockwise.

In general, the definition of the classes in the frequency histogram is flexible. The general purpose of a frequency histogram is very much the same as that of a stem and leaf diagram, to provide a graphical display that gives a sense of data distribution across the range of values that appear. **Relative frequency histograms**

In our previous example of the exam scores in a statistics class, five students scored in the 80s.

The number 5 is the *frequency* of the group labelled “80s.” Since there are 30 students in the entire statistics class, the proportion who scored in the 80s is $5/30$.

The number $5/30$, which could also be expressed as $0.1667 = 0.16670.16$, or as $16.67\%$, is the *relative frequency* of the group labelled “80s.”

Every group (the 70s, the 80s, and so on) has a relative frequency. We can thus construct a diagram by drawing for each group, or class, a vertical bar whose length is the relative frequency of that group.

For example, the bar for the 80s will have length $5/30$ unit, not 5 units. The diagram is a *relative frequency histogram* for the data. It is exactly the same as the frequency histogram except that the vertical axis in the relative frequency histogram is not frequency but relative frequency.
The same procedure can be applied to any collection of numerical data. Classes are selected, the relative frequency of each class is noted, the classes are arranged and indicated in order on the horizontal axis, and for each class a vertical bar, whose length is the relative frequency of the class, is drawn. The resulting display is a relative frequency histogram for the data. A key point is that now if each vertical bar has width 1 unit, then the total area of all the bars is 1 or 100%.

Although the histogram and the Frequency Histogram have the same appearance, the relative frequency histogram is more important for us. It is the relative frequency histogram that is used repeatedly to represent data.

To see why this is so, reflects on what it is that you are actually seeing in the diagrams that quickly and effectively communicates information to you about the data. It is the relative sizes of the bars.

The bar labelled “70s” takes up 1/3 of the total area of all the bars, and although we may not think of this consciously, we perceive the proportion 1/3 in the figures, indicating that a third of the grades were in the 70s.
The relative frequency histogram is important because the labelling on the vertical axis reflects what is important visually: the relative sizes of the bars.

When the size $n$ of a sample is small only a few classes can be used in constructing a relative frequency histogram. Such a histogram might look something like the one in panel (a) of sample size and relative frequency histograms.

If the sample size $n$ were increased, then more classes could be used in constructing a relative frequency histogram and the vertical bars of the resulting histogram would be finer, as indicated in panel (b) of Sample Size and Relative Frequency Histograms.

For a very large sample the relative frequency histogram would look very fine, like the one in (c) of sample size and relative frequency histograms.

If the sample size were to increase indefinitely then the corresponding relative frequency histogram would be so fine that it would look like a smooth curve, such as the one in panel (d) of sample size and relative frequency histograms.
It is common in statistics to represent a population or a very large data set by a smooth curve. It is good to keep in mind that such a curve is actually just a very fine relative frequency histogram in which the exceedingly narrow vertical bars have disappeared.

Because the area of each such vertical bar is the proportion of the data that lies in the interval of numbers over which that bar stands, this means that for any two numbers \( a \) and \( b \), the proportion of the data that lies between the two numbers \( a \) and \( b \) is the area under the curve that is above the interval \((a, b)\) in the horizontal axis.

This is the area shown as ‘A Very Fine Relative Frequency Histogram’. In particular, the total area under the curve is 1, or 100%.

**Scatterplots**

We now consider two dimensional data. The values of the first variable \( x_1; x_2; \ldots; x_n \) are assumed known and in an experiment and are often set by the experimenter. This variable is called the independent, explanatory, predictor, descriptor or input variables and in a two dimensional scatterplot of the data display its values on the horizontal axis.

The values \( y_1; y_2, \ldots, y_n \), taken from observations with input \( x_1; x_2; \ldots; x_n \) are called the dependent, response or target variable and its values are displayed on the vertical axis.

In describing a scatterplot, take into consideration:

a). the form
   - linear
- curved relationships
- clusters

b). the direction
- a positive or negative association
- the strength of the aspects of the scatterplot.

**Activity 5**

1. Describe one difference between a frequency histogram and a relative frequency histogram.

**Solution**

The vertical scale on the frequency histogram is the frequencies and on the relative frequency histogram is the relative frequencies.

2. Construct a stem and leaf diagram, a frequency histogram, and a relative frequency histogram for the following data set. For the histograms use classes 51–60, 61–70, and so on: 69, 70, 93, 53, 92, 85, 75, 70, 68, 88, 76, 70, 77, 85, 82, 82, 80, 96, 100, 85.

**Solution**

<table>
<thead>
<tr>
<th>5</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>8 9</td>
</tr>
<tr>
<td>7</td>
<td>0 0 0 5 6 7</td>
</tr>
<tr>
<td>8</td>
<td>0 2 3 5 5 5 8</td>
</tr>
<tr>
<td>9</td>
<td>2 3 6</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
</tr>
</tbody>
</table>

Frequency and relative frequency histograms are similarly generated.
UNIT 6 BASIC PROBABILITY: LEARNING, TEACHING AND APPLYING
UNIT INTRODUCTION
Dear Students,
In the last unit, we learned basic concepts of statistics. In this unit we are going to learn the basic concepts of probability, namely:

1. Fundamental counting principles
2. Basic concepts of probability
3. Dependent events with replacements and without replacements
4. Dependent events with conditional probabilities and tree diagrams
5. Conditional probability and Bayes theorem
6. Applications of permutations and combinations in probability

UNIT 6 SECTION 1 FUNDAMENTAL COUNTING PRINCIPLES
Introduction
Dear Students,
Mathematics began with counting. Initially, fingers, pebbles, sticks and bottletops were used to help with counting, but these are only practical for small numbers. What happens when a large number of items must be counted? Therefore, this section focuses on how to use mathematical techniques to count different assortments of items. Our learning indicators are:
• explain counting with repetition and without repetition
• explain and apply the principles of permutations and combinations

The Fundamental Counting Principle
In counting, the use of lists, tables and tree diagrams is only feasible for events with a few outcomes. When the number of outcomes grows, it is not practical to list the different possibilities and the fundamental counting principle is used instead.
The fundamental counting principle states that if there are $n(A)$ outcomes in event A and $n(B)$ outcomes in event B, then there are $n(A) \times n(B)$ outcomes in event A and event B combined.

**Example**

Given $A = \{1,2,3,4,5,8\}$ and $B = \{5,7,9\}$

Then $n(A) = 6$ and $n(B) = 3$

Therefore, $n(A) \times n(B) = 6 \times 3 = 18$

**Choices without Repetition**

If there are $n_1$ possible outcomes for event A and $n_2$ outcomes for event B, then the total possible number of outcomes for both events is $n_1 \times n_2$.

This can further be generalised to k events, where k is the number of events. The total number of outcomes for k events is:

$$n_1 \times n_2 \times n_3 \times … \times n_k.$$

**Examples**

1. What is the total number of possible outcomes when a die is rolled and then a coin is tossed?

**Solution**

The roll of a die has six possible outcomes (1;2;3;4;5; 6) and the toss of a coin, 2 outcomes (heads or tails).

The sample space (total possible outcomes) can be represented as follows:

$S=\{(1;H); (1;T); (2;H); (2;T);(3;H);(3;T);(4;H);(4;T);(5;H);(5;T);(6;H);(6;T)\}$

$S=\{(1;H); (2;H); (3;H); (4;H);(5;H);(6;H); (1;T);(2;T);(3;T);(4;T);(5;T);(6;T)\}$

Therefore there are 12 possible outcomes.

In other words, $n(A) \times n(B) = 6 \times 2 = 12$

2. A restaurant has a 4-piece lunch special which consists of a sandwich, soup, dessert and drink for 50.00 Ghana cedis. They offer the following choices for:

- **Sandwich**: chicken mayonnaise, cheese and tomato, tuna mayonnaise, ham and lettuce
- **Soup**: tomato, chicken noodle, vegetable
- **Dessert**: ice-cream, piece of cake
Drink: tea, coffee, Coke, Fanta, Sprite

a). How many parts are there in the meal?

b). How many possible meals are there?

Solution

a). There are 4 parts: sandwich, soup, dessert and drink.

b). The possible number of meals:

<table>
<thead>
<tr>
<th>Meal component</th>
<th>Sandwich</th>
<th>Soup</th>
<th>Dessert</th>
<th>Drink</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of choices</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>

Using the fundamental counting principle, the possible different meals are:

\[ n_1 \times n_2 \times n_3 \times n_4 = 4 \times 3 \times 2 \times 5 = 120 \]

So there are 120 possible meals.

3. If a coin is flipped three times, what is the total number of different results?

Solution

Each time a coin is flipped, there are two possible outcomes, namely heads or tails. The coin is flipped 3 times. We that there is a total of 8 different possible outcomes.

NB: Drawing a tree diagram is possible to draw for three different coin flips, but as soon as the number of events increases, the total number of possible outcomes increases to the point where drawing a tree diagram is impractical.

4. What is the total if we flip a coin six times.

Solution

In this case, using the fundamental counting principle is a far easier option.

We know that each time a coin is flipped that there are two possible outcomes.
So if we flip a coin six times, the total number of possible outcomes is equivalent to multiplying 2 by itself six times.
Therefore, $2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^6 = 64$

5. Diana packed 2 skirts, 4 blouses, and 1 sweater to go to school. She will need to choose a skirt and a blouse for each outfit and decide whether to wear the sweater depending on the weather conditions. Find the total number of possible outfits.

**Solution**
To find the total number of outfits, we need to find the product of the number of skirt options, the number of blouse options, and the number of sweater options as shown below:

<table>
<thead>
<tr>
<th>Number of Skirt options</th>
<th>Number of Blouse options</th>
<th>Number of Sweater options</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>4</td>
<td>2</td>
</tr>
</tbody>
</table>

Hence, $n_1 \times n_2 \times n_3 = 4 \times 2 \times 2 = 16$

There are 16 possible outfits.

**Choices with Repetition**
If we have the letters A, B, C, and D and we wish to discover the number of ways of arranging them in three-letter patterns if repetition is allowed, such as ABA, DCA, BBB, etc., we will find that there are 64 ways.

This is because for the first letter of the pattern, we can choose any of the four available letters, for the second letter of the pattern, we can choose any of the four letters, and for the final letter of the pattern, we can choose any of the four letters.

Multiplying the number of available choices for each letter in the pattern gives the total available arrangements of letters: $4 \times 4 \times 4 = 4^3 = 64$

This allows us to formulate the following: when you have $n$ objects to choose from and you choose from them $r$ times, then the total number of possibilities is $n \times n \times n \times \ldots \times n (r \text{ times}) = n^r$.

**Examples**
1. A school plays a series of 6 soccer matches. For each match there are 3 possibilities: a win, a draw or a loss. How many possible results are there for the series?

**Solution**

Step 1: We will determine how many outcomes to choose from for each event:

There are 3 outcomes for each match: win, draw or lose \(n\).

Step 2: We will determine the number of events:

There are 6 matches, so the number of events is \(r\).

Step 3: We will determine the total number of possible outcomes

There are 3 possible outcomes for each of the 6 events. Therefore, the total number of possible outcomes for the series of matches is \(n^r = 3^6 = 729\)

**Permutations of \(n\) Distinct Objects**

The multiplication principle can be used to solve a variety of problem types. One type of problem involves placing objects in order.

Our daily applications are arranging letters into words and digits into numbers, lining up for photographs, and decorating rooms. An ordering of such objects is called a permutation.

To solve permutation problems, it is often helpful to draw line segments for each option to enables us determine the number of each option to multiply. For instance, suppose we have four paintings, and we want to find the number of ways we can hang three of the paintings in order on the wall. We can draw three lines to represent the three places on the wall.

**Procedure for a permutation**

- Determine how many options there are for the first situation.
- Determine how many options are left for the second situation.
- Continue until all of the spots are filled.
- Multiply the numbers together.

**Examples**

1. At a swimming competition, nine swimmers compete in a race.
   
   a. How many ways can they place first, second, and third?
b. How many ways can they place first, second, and third if a swimmer named Aba wins first place? (Assume there is only one contestant named Aba.)
c. How many ways can all nine swimmers line up for a photo?

Solution

a. Draw lines for each place.

<table>
<thead>
<tr>
<th>Options for first place</th>
<th>x</th>
<th>Options for second place</th>
<th>x</th>
<th>Options for third place</th>
</tr>
</thead>
</table>

There are 9 options for first place. Once someone has won first place, there are 8 remaining options for second place. Once first and second place have been won, there are 7 remaining options for third place.

Therefore, \(9 \times 8 \times 7 = 504\)

Multiply to find that there are 504 ways for the swimmers to place.

b. Draw lines for describing each place.

<table>
<thead>
<tr>
<th>Options for first place</th>
<th>x</th>
<th>Options for second place</th>
<th>x</th>
<th>Options for third place</th>
</tr>
</thead>
</table>

We know Aba must win first place, so there is only 1 option for first place. There are 8 remaining options for second place, and then 7 remaining options for third place.

\[
1 \times 8 \times 7 = 56
\]

Multiply to find that there are 56 ways for the swimmers to place if Ariel wins first.

c. Draw lines for describing each place in the photo.

There are 9 choices for the first spot, then 8 for the second, 7 for the third, 6 for the fourth, and so on until only 1 person remains for the last spot.

\[
9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362,880
\]

There are 362,880 possible permutations for the swimmers to line up.

**Permutations of \(n\) Distinct Objects Using a Formula**

If there are so many numbers to multiply as in example 1c above, it is inconvenient to use the Multiplication Principle.
There are two common notations for this permutation, namely \( P(n, r) \) or \( nPr \), where \( n \) is the set of objects and we want to choose \( r \) objects from the \( n \) set in order.

The general formula is as \( nPr = \frac{n!}{(n-r)!} \)

**Procedures**

- We begin by finding \( n! \), the number of ways to line up all \( n \) objects.
- We then divide by \( (n-r)! \).
- We cancel out the \( (n-r)! \) items that we do not wish to line up.
- We then obtain the actual number we wish to line up

**Example**

Imagine that six students has formed a study group, and they need to elect a president, a vice president, and a treasurer. How many possible ways can they do this exercise fairly to everyone in the group.

**Solution**

Any of the six students could be elected president, any one of the five remaining students could be elected vice president, and any of the remaining four students could be elected treasurer.

We know that the number of ways this may be done is \( 6 \times 5 \times 4 = 120 \).

However, in the using factorials, we get the same result as: \( nPr = \frac{6!}{3!} = \frac{6 \times 5 \times 4 \times 3!}{3!} = 120 \)

NB: The formula stills works if we are choosing all \( n \) objects and placing them in order. In that case we would be dividing by \( (n-n)! \) or \( 0! \), which is 1. So the number of permutations of \( n \) objects taken \( n \) at a time is \( n! \).

**Example**

A mathematics professor is creating an exam of 9 questions from a test bank of 12 questions. How many ways can she select and arrange the questions?

**Solution**

Substitute \( n=12 \) and \( r=9 \) into the permutation formula and simplify.
Combinations of n Distinct Objects Using the Formula

So far, we have looked at problems asking us to put objects in order. There are many problems in which we want to select a few objects from a group of objects, but we do not care about the order. When we are selecting objects and the order does not matter, we are dealing with combinations.

A selection of $r$ objects from a set of $n$ objects where the order does not matter can be written as $C(n, r)$ or $nCr$, just as with permutations,

In this case, the general formula is $C(n, r) = \frac{n!}{(n-r)!r!}$.

- Identify $n$ from the given information.
- Identify $r$ from the given information.
- Replace $n$ and $r$ in the formula with the given values.
- Evaluate the formula

Examples

1. How many ways can a painter select 3 colours out of 4 for a building if order is not considered?

Solution

If we do not care about the order, we would expect a smaller number because selecting paintings 1,2,3 would be the same as selecting paintings 2,3,1.

To find the number of ways to select 3 of the 4 paintings, disregarding the order of the paintings, divide the number of permutations by the number of ways to order 3 paintings.

There are $3! = 3 \times 2 \times 1 = 6$ ways to order 3 paintings. There are 24 or 4 ways to select 3 of the 4 paintings. This is because every time we are selecting 3 paintings, we are not selecting 1 painting.

There are 4 paintings we could choose not to select, so there are 4 ways to select 3 of the 4 paintings.

2. UEW food restaurant offers five side dish options. Your meal comes with two side dishes.

a). How many ways can you select your side dishes?

b). How many ways can you select 33 side dishes?

Solution

a). We want to choose 2 side dishes from 5 options.
\[ C(n, r) = \frac{n!}{(n-r)!r!} = \frac{5!}{(5-2)!2!} = 10 \]

b). We want to choose 33 side dishes from 55 options.
\[ C(n, r) = \frac{n!}{(n-r)!r!} = \frac{55!}{(55-33)!33!} = 10 \]

Using Combinations to Calculate the Number of Subsets of a Set
We have looked only at combination problems in which we chose exactly \( r \) objects. In some cases, we want to consider choosing every possible number of objects.
We know that a set containing \( n \) distinct objects has \( 2^n \) subsets.
Therefore, the possible combinations are \( \sum_{r=0}^{n} nCr \).

Examples
1. A pizza restaurant offers 5 toppings. Any number of toppings can be ordered. How many different pizzas are possible?

Solution
Here, we need to consider pizzas with any number of toppings.
Note that there is \( C(5,0) = 1 \) way to order a pizza with no toppings. There are \( C(5,1) = 5 \) ways to order a pizza with exactly one topping.

If we continue this process, we get: \( C(5,0) + C(5,1) + C(5,2) + C(5,3) + C(5,4) + C(5,5) = 32 \)
There are 32 possible pizzas. This result is equal to \( 2^5 \).

We are presented with a sequence of choices. For each of the \( n \) objects we have two choices: include it in the subset or not. So for the whole subset we have made \( n \) choices, each with two options. So there are a total of \( 2 \times 2 \times 2 \times \ldots \times 2 \) possible resulting subsets.

Thus we start all the way from the empty subset, which we obtain when we say “no” each time, to the original set itself, which we obtain when we say “yes” each time.

2. A restaurant offers butter, cheese, chives, and sour cream as toppings for a baked potato. How many different ways are there to order a potato?

Solution
We are looking for the number of subsets of a set with 4 objects. Substitute \( n = 4 \) into \( n! \).
We have \( 2^n = 2^4 = 16 \)
There are 16 possible ways to order a potato.
Finding the Number of Permutations of \( n \) Non-Distinct Objects

For example, suppose there is a sheet of 12 stickers. If all of the stickers were distinct, there would be 12! ways to order the stickers. However, if 4 of the stickers are identical stars, and 3 are identical moons, then all the objects are not distinct, and many of the 12! Permutations may be duplicated.

Therefore, the general formula for this situation is

\[
P(n, r) = \frac{n!}{r_1!r_2!...r_k!}
\]

In this example, we need to divide by the number of ways to order the 4 stars and the ways to order the 3 moons to find the number of unique permutations of the stickers.

There are 4! ways to order the stars and 3! ways to order the moon.

\[
P(n, r) = \frac{n!}{r_1!r_2!...r_k!} = \frac{12!}{4!3!} = 3,326,400
\]

Example

Find the number of rearrangements of the letters in the word DISTINCT.

Solution

There are 8 letters. Both I and T have repeated 2 times.

Substitute \( n=8, r_1=2, \) and \( r_2=2 \) into the formula

\[
P(n, r) = \frac{n!}{r_1!r_2!...r_k!} = \frac{8!}{2!2!} = 10,080
\]

Summary

- In this section, we have learned permutations and combinations.
- We learned that permutation is used to select objects with order and combination s used to select objects without order.
- We have also learned that the objects can be distinct or non-distinct.

UNIT 6 SECTION 2 BASIC CONCEPTS OF PROBABILITY

Introduction

Dear Students,

Probability began with experiments, guessing outcomes and computing likely events from the outcomes. Initially, we start with dice, coins, and cards, ages or birthdays of students to complex issues in data. What happens when a data set is large? Then we need complex mathematical techniques in probability distributions.
Here, our learning indicators are:

- explain the basic concept of experiments, outcomes, equally likely, sample space, events, and compound events.
- explain and apply the principles of addition and multiple rules of probability

**Experiment**
An experiment is a process by which an outcome is obtained. Therefore, a random experiment is a process or action whose outcome is not determined.

**Examples**
1. rolling a die
2. tossing a coin
3. selecting a card
4. selecting a colour
5. throwing a ball
6. measuring ages or height or weights of pupils
7. choosing a number
8. writing an English vowel
9. drawing a shape

**Outcome**
An outcome is a single result from a measurement of an experiment.

**Examples**
1. one of head or tail from a coin
2. one of 1,2,3,4,5 or 6 from a die
3. one of heart, spade, diamond or culet of a deck of cards
4. one of ‘a’, ‘e’, ‘I’, ‘o’, or ‘u’ of the English vowels
5. one 2,3,5,7,11,13,17, or 19 of prime numbers less than 20

**Equally likely outcomes**
Two outcomes of a random experiment are said to be equally likely, if upon performing the experiment a (very) large number of times, the relative occurrences of the two outcomes turn out to be equal.

**Examples**
1. For a perfectly fair coin, the relative occurrences of $H$ and $T$ for a very large number of tosses $N$ will be equal (as $n$ goes to infinity, we will come closer to a perfect equality).
2. For an unbiased, unloaded die, each of the six outcomes is equally likely.
3. For an unbiased, unloaded deck of playing cards, each of the four outcomes (hearts, clubs, diamonds and spades) is equally likely.

**Sample Space**
In probability, the set of all possible outcomes is called the *Sample Space*. We use $S$ to represent the sample space.

In terms of the language of sets, a sample space is a universal set and an outcome is an element of the universal set.

**NB**: The number of all possible outcomes may be finite, infinite or continuous.

**Examples**
1. The sample space for the experiment of tossing a coin once is $S = \{H, T\}$ because there are only two possible outcomes, Heads or Tails.
2. The sample space for the experiment of tossing a standard die is $S = \{1, 2, 3, 4, 5, 6\}$ because these are the only six possible outcomes.

2. The sample space for the experiment of selecting a playing from a standard deck of cards is $S = \{1, 2, 3, \ldots 52\}$ because these are the only 52 possible outcomes.

3. The set of prime numbers $S = \{2, 3, 5, 7, 11, 13, 17, 19\ldots\}$

**An Event**

It is a particular result or set of results amongst the possibilities in the sample space.

**Examples**

1. obtaining odd numbers from tossing a die
2. obtaining head from throwing a coin
3. obtaining a heart from selecting cards

**The concept of probability**

Probability is the likelihood that an event will occur. It is written as a fraction with the number of favourable outcomes as the numerator and the total number of outcomes as the denominator. Favourable just means that a particular outcome is what you are curious about, not that it is necessarily positive.

Probability can be used to determine many things, from the likelihood that you will win the jackpot in the lottery to the likelihood that a baby will be born with a certain birth defect and anything in between. Probability is used extensively in the sciences, investing, weather reporting and many other areas.

**Three Definitions of Probability**

Probability can be expressed as a percentage, a fraction, a decimal, or a ratio.

**A. Classical definition**

If there are a finite number of possible outcomes of an experiment, all equally likely and mutually exclusive, then the probability of an event (A) is the number of outcomes favourable to the event, divided by the total number of possible outcomes.

That is, $P(A) = \frac{n(A)}{n(S)}$

**Examples**

1. A fair die is rolled once. What is the probability that the outcome is more than 4?

**Solution**
\[ S = \{1,2,3,4,5,6\} \text{ and } A = \{5,6\} \]

Thus \( n(S) = 6 \) and \( n(A) = 2 \)

Therefore \( (A) = \frac{n(A)}{n(S)} = \frac{2}{6} = \frac{1}{3} \)

\[ \text{B. Statistical or relative frequency definition} \]

The probability of an event denotes the relative frequency of occurrence of that event in the long run. For instance, the probability of a newborn infant being female is estimated to be about .51 in Ghana. This is also called the frequentist definition and is the one in common use.

But it is not a fully satisfactory definition. What does \textit{in the long run} mean? And what about situations in which the experiment cannot be repeated indefinitely under identical conditions, even in principle?

\[ \text{C. The axiomatic approach} \]

A mathematically precise approach the axiomatic definition of probability, which incorporates both classical and relative frequency. It begins with some abstract terms and then defines a few basic axioms on which an elaborate logical structure can be built using the mathematical theories of sets and measure.

In other words, it is a type of probability that has a set of axioms (rules) attached to it. For example, you could have a rule that the probability must be greater than 0\%, that one event must happen, and that one event cannot happen if another event happens.

It could also states that a probability is a number between zero and one, but nothing is specified about how to assign it. Assignment may be based on a model or on experimental data.

Developments are valid if they follow from the axioms, independent of any correspondence to phenomena of the physical world. The following statements represent the axioms of probability.

\textbf{Axioms of probability}

Let \( S \) be a finite sample space, \( A \) an event in \( S \). We define \( P(A) \), the probability of \( A \), to be the value of an additive set that satisfies the following three conditions

\textit{Axiom 1:} \( 0 \leq PA \leq 1 \) for each event \( A \) in \( S \) (probabilities are real numbers between 0 and 1 inclusive).
Axiom 2: \( (S) = 1 \) (the probability of some event occurring from \( S \) is unity).

Axiom 3: If \( A \) and \( B \) are mutually exclusive events in \( S \), then \( (A \cup B) = P(A) + P(B) \) (the probability function is an additive set function).

Axiom 4: If \( A \) is an event in \( S \), then \( P(\overline{A}) = 1 - P(A) \), where \( \overline{A} \) is the complement of \( A \).

**Compound Events**

A compound event is one in which there is more than one possible outcome. Determining the probability of a compound event involves finding the sum of the probabilities of the individual events and, if necessary, removing any overlapping probabilities.

**Examples of compound events**

Let's take a look at some examples.

1) What is the probability that you will roll a five using a 6-sided die?

The favourable outcome is rolling a five, and that can only occur once using one die. The total number of outcomes is six, since the die is 6-sided.

So the probability of rolling a five is \( \frac{1}{6} \).

2) What is the probability that you will pull a heart out of a standard deck of cards?

The favourable outcome would be pulling a heart and there are 13 of them in a standard deck. The total number of outcomes is 52 because there are 52 cards in a standard deck.

The probability of pulling a heart is \( \frac{13}{52} \) or \( \frac{1}{4} \).

**Types of Compound Events**

A compound event is an event with two or more favourable outcomes. There are three types of compound events and determining the probability for each is different. First, let’s talk about an exclusive compound event.

**A. An exclusive compound events**

An exclusive compound event in one in which the multiple events do not overlap. The method for determining the probability of this type of compound event is to add together the probabilities of each event.
Compound events that cannot happen at the same time are called *mutually exclusive* events. For example, a number cannot be both even and odd or you cannot have picked a single card from a deck of cards that is both a ten and a jack.

NB: *Mutually inclusive* events, however, can occur at the same time. For example, a number can be both less than 5 and even or you can pick a card from a deck of cards that can be a club and a ten.

When finding the probability of events occurring at the same time, there is a concept known as the “double counting” feature. It happens when the intersection is counted twice.

But if the events are mutually exclusive events, then \( P(A \text{ and } B) = 0 \), because they cannot happen at the same time. To find the probability of either mutually exclusive event \( A \) or \( B \) occurring, we say that \( P(A \text{ or } B) = P(A) + P(B) \)

If the events are mutually inclusive, then the probability of one or the other mutually inclusive event is to add the individual probabilities and subtract the probability they occur at the same time. That is, \( P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B) \).

Note that finding the probability of one or the other *mutually exclusive* event is the same as the formula for finding the probability of one or the other *mutually inclusive* event except \( P(A \text{ and } B) = 0 \) in the case of the mutually exclusive events.

**Examples**

1). What is the probability of rolling either a two or a four using one 10-sided die?

**Solution**

The probability of rolling a two is \( \frac{1}{10} \) and the probability of rolling a four is \( \frac{1}{10} \).

So, the compound probability is: \( P(C) = \frac{1}{10} + \frac{1}{10} = \frac{2}{10} \) or \( \frac{1}{5} \)

2). What is the probability of pulling any face card or a three of clubs from a standard deck of cards?

**Solution**

The probability of getting a face card is \( \frac{12}{52} \) and the probability of getting a three of clubs is \( \frac{1}{52} \).

So the compound probability is \( P(C) = \frac{12}{52} + \frac{1}{52} = \frac{13}{52} \) or \( \frac{1}{4} \)

3). What is the probability of drawing a black card or a ten in a deck of cards?
Solution

There are 4 tens in a deck of cards $P(\text{tens}) = \frac{4}{52}$

There are 26 black cards $P(\text{black}) = \frac{26}{52}$

There are 2 black tens $P(\text{black and ten}) = \frac{2}{52}$

$P(\text{black or ten}) = \frac{4}{52} + \frac{26}{52} - \frac{2}{52} = \frac{30}{52} - \frac{2}{52} = \frac{28}{52} = \frac{7}{13}$

4). Using a standard deck of cards, find the probability of:

a. $P(\text{jack or a king})$

b. $P(\text{jack or a spade})$

Solutions

a. $P(\text{jack}) = \frac{4}{52}$ (there are 4 jacks in a deck of 52 cards)

$P(\text{kings}) = \frac{4}{52}$ (there are 4 kings in a deck of 52 cards)

$P(\text{jack or a king}) = \frac{4}{52} + \frac{4}{52} = \frac{8}{52} = \frac{2}{13}$ (since these are mutually exclusive)

b. $P(\text{jack or a spade})$

$P(\text{jack}) = \frac{4}{52}$

$P(\text{spade}) = \frac{13}{52}$

$P(\text{jack and spade}) = \frac{1}{52}$ (there is one jack that is also a spade)

$P(\text{jack or a spade}) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

B. Independent Compound Events

In the language of mathematics, we can say that all those events whose probability does not depend on the occurrence or non-occurrence of another event are independent events. For example, if we have two coins and if we flip these two coins together, then each one of them can either turn up a head or a tail and the probability of one coin turning either a head or a tail is totally independent of the probability of the other coin turning up a head or a tail.

The probability of an independent event in the future is not dependent on its past. For example, if you toss a coin three times and the head comes up all the three times, then what is the probability of getting a tail on the fourth try? The answer is simply 1/2.

Example
1. Out of the following examples, which represents an independent event?
A). The probability of drawing an Ace from a well-shuffled pack of 52 cards, twice.
B). Probability of drawing a King from a pack of 52 cards and an Ace from another well-shuffled pack of 52 cards.
C). Two queens which we draw out of a well-shuffled pack of 52 cards.
D). All of the above events are examples of independent events.

Solution
In option A), the two of events are drawing an ace and then drawing another ace. When we draw the first ace, we have one event in our favour and 52 in total. So the probability is 1/52. For the second draw, there is 1 less card in the deck, so these events in which we have only one pack of cards cannot be independent events.

In option B), Probability of drawing a King from a pack of 52 cards and an Ace from another well-shuffled pack of 52 cards are independent events.

In option C), the two of events are drawing a queen and then drawing another queen. When we draw the first queen, we have one event in our favour and 52 in total. So the probability is 1/52. For the second draw, there is 1 less card in the deck, so these events in which we have only one pack of cards cannot be independent events.

In option D), not true that of the above events are independent events.

The Rule of products is only applicable to the events that are independent of each other. The product gives the total probability of such events. In other words, the probability of all such events occurring is what we get from the product of probabilities.

C. Nonmutually exclusive and complementary events
The Rule of Complements defines the probability of the complement of an event in terms of the probability of the original event. Consider event A defined over the sample space S. The complement of set A, denoted by \( \overline{A} \), is a subset, which contains all outcomes, which do not belong to A.

In other words \( P(A) + P(\overline{A}) = 1 \) or \( P(\overline{A}) = 1 - P(A) \).

Example
1. Find the probability of the event of getting a total of less than 12 in the experiment of throwing a die twice.
Solution

Let A be the event of getting a total 12.

The event of getting a total of less than 12 is the complement of A

But \( P(A) = \frac{1}{36} \)

Hence \( P(\overline{A}) = 1 - \frac{1}{36} = \frac{35}{36} \)

2. In a sample of 55 people, 28 have brown hair and 22 have blue eyes. 5 of them have neither brown hair nor blue eyes. What is the probability that a random person from the sample has at least one of these features?

Solution

Our compound events here are “brown hair”, and “blue eyes.” To find the probability that a person chosen at random from the sample of 55 people has at least one of these features, we can simply note that since 5 of the 55 have neither feature, all the rest must have at least one, that is, that 55−5=50 of the 55 have either brown hair or blue eyes or both.

The probability that a person chosen at random has either brown hair, blue eyes, or both is, therefore, \( P(\cup) = \frac{50}{55} = \frac{10}{11} \). Brown hair + Blue eyes

In essence, we have used our total probability rule to calculate to know that, for event \( A \), \( (A) = 1 - P(A^{-}) \) and neither brown hair nor blue eyes \( P(\cup) = 1 - P(\cup) = 1 - \frac{5}{55} = \frac{50}{55} = \frac{10}{11} \).

Hence, as noted, the probability that a person chosen at random from the sample has at least one of the features “brown hair” and “blue eyes” is 10/11.

Summary

In this section that we have learned that an experiment is a process by which an outcome is obtained, an outcome is a single result from a measurement of an experiment and an event is a particular result or set of results amongst the possibilities in the sample space.

We have also learned that the Axioms of probability are \( 0 \leq PA \leq 1 \) (probabilities are real numbers between 0 and 1 inclusive), \( P(S) = 1 \) (the probability of some event occurring from \( S \) is unity), if \( A \) and \( B \) are mutually exclusive events in \( S \), then \( P(A \cup B) = PA + PB \) (the probability function is an additive set function), and if \( A \) is an event in \( S \), then \( P(\overline{A}) = 1 - P(A) \), where \( (\overline{A}) \) is the complement of \( A \).
Again, we have learned that a compound event is an event with two or more favourable outcomes and there are three types of compound events and determining the probability for each is different. These are mutually exclusive or mutually inclusive, independent or dependent, and nonmutually exclusive or complementary events.

UNIT 6 SECTION 3 DEPENDENT EVENTS WITH REPLACEMENTS AND WITHOUT REPLACEMENTS

Introduction
In this unit 6 section 2, we learned the basic concepts of an experiment, an outcome and an event. We also learned the axioms of probability, compound events, mutually exclusive or mutually inclusive, independent or dependent, and nonmutually exclusive or complementary events. In this section, we are going to learned basic concepts in dependents events with replacements and events without replacements and the total probability rule.
Suppose you flip a coin and roll a die at the same time. These are compound events. What is the probability you will flip a head and roll a four? These events are independent. Rolling a die has no effect on flipping a coin, and so replacing the dice or not does not affect the probability of the coin.

However, suppose you randomly draw a card from a standard deck and then randomly draw a second card without replacing the first. The second probability is now different from the first and depends on the first.

If the second card is replaced, then the probabilities of the two events will be the same. However, if the second card is not replaced, then the probability of the two will differ since the total deck of cards will be reduced accordingly.

**Dependent Events with replacements and events without replacements**
To find the probability of two dependent events, multiply the probability of the first event by the probability of the second event, after the first event occurs. \( P(A \text{ and } B) = P(A) \times P(B \text{ following } A) \)

**Example**

1. Two cards are drawn from a deck of cards. Let:

   A: 1st card is a club
   
   B: 1st card is a 7
   
   C: 2nd card is a heart

   Find the following probabilities:

   a. \( P(A \text{ or } B) \)

   A club or a 7 can be picked at the same time so these are mutually inclusive events. You can use the formula from above.

   \[
   P(A \text{ or } B) = \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}
   \]

   b. \( P(B \text{ or } A) \)

   c. A club and a 7 can be picked at the same time so these are mutually inclusive events. You can use the formula from above.

   \[
   P(B \text{ or } A) = \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}
   \]

   d. \( P(A \text{ and } C) \)

   Picking a club on the first card and a heart on the second card are dependent events so you need to multiply the probability of A by the probability of C following A.

   \[
   P(A \text{ and } C) = \frac{13}{52} \times \frac{13}{51} = \frac{169}{2652}
   \]

2. A bowl contains 12 red marbles, 5 blue marbles and 13 yellow marbles. Find the probability of drawing a blue marble and then drawing a yellow marble.
Solution

Let A=blue marble chosen 1st and B=yellow marble chosen 2nd. The total number of marbles in the bowl is $12+5+13=30$.

$P(A)=\frac{5}{30}$, and $P(B)=\frac{13}{29}$ (Remember, one marble has been removed).

$P(A \text{ and } B)=\frac{5}{30}\times\frac{13}{29}$ or $P(A \cap B)=\frac{5}{30}\times\frac{13}{29}=\frac{65}{870}$ or $P(A \cap B)=\frac{13}{174}$

3. $A$ and $B$ are independent events, where $(A)=13$ and $P(B)=25$. What is the probability that events $A$ and $B$ both occur?

Solution

Given that events $A$ and $B$ are independent, the probability that they both occur is $(A \cap B)=P(A)\times P(B)=\frac{1}{3}\times\frac{25}{30}=\frac{2}{15}$.

4). A bag contains 22 red balls and 9 green balls. One red ball is removed from the bag and then a ball is drawn at random. Find the probability that the drawn ball is red.

Solution

To find the probability of drawing a second red ball from the bag, we note first that since there are 22 red balls and 9 green ones, there are $22+9=31$ balls in total.

The probability of drawing a red ball from the bag on our first pick is, therefore, $P(R)=\frac{22}{31}$.

That is, red number of red balls total number of balls

Since we are keeping the first ball out of the bag, there is one less ball in the bag in total, so there are now 30 balls in the bag. And of those 30, there is one less red since the ball we took out was red.

Hence, the number of red balls is now 21.

To help us work out the probability of drawing a second red ball from the bag, we can illustrate the probability of taking a second red ball having not replaced the first is found by multiplying the probabilities of the “first ball red” and “second ball red”: $P(\cap) = \frac{22}{31}\times\frac{21}{30} = \frac{77}{155} \approx 0.497$.

The probability that the second ball drawn is red is therefore 0.497. We can say that there is approximately a 50% chance of choosing two consecutive red balls (since $0.497\times100%=49.7\%$).
Note that we have actually used the formula for compound dependent events: $(A \cap B) = P(A|B) \times P(B)$.

The probability $21/30$ is the conditional probability of selecting a red ball given that a red ball has already been taken from the bag:

5. Two coins are flipped simultaneously. What is the probability of getting heads on either of these coins?

**Solution**
First thing that you realise is that these are independent events. Once you do that, move on to find the probability of each individual event.

Let us call the first coin toss as E and the second coin toss as F. Therefore we can write: $P(E) = 1/2$ i.e. probability of getting a head on the first coin toss = $1/2$.

Similarly, the probability of getting a head on the second coin’s toss = $1/2$. In other words, we can write that $P(F) = 1/2$.

Now we have to calculate the probability of both these events happening together. Hence we use the rule of the product. If $P$ is the probability of some event and $Q$ is the probability of another event, then the probability of both $P$ and $Q$ happening together is $P \times Q$.

Hence the probability that either of the two coins will turn up a head = $1/2 \times 1/2 = 1/4$

6. A die is cast twice and a coin is tossed twice. What is the probability that the die will turn a 6 each time and the coin will turn a tail every time?

**Solution**
Each time the die is cast, it is an independent event. The probability of a getting a 6 is = $1/6$. So the probability of getting a 6 when the die is cast twice = $1/6 \times 1/6 = 1/36$.

Similarly the probability of getting a tail in two flips that follow each other (are independent) = $(1/2) \times (1/2) = 1/4$.

Therefore as the two events i.e. casting the die and tossing the coin are independent, and the probability of both the events = $(1/36) \times (1/4) = 1/144$.

7. If one has three dice what is the probability of getting three 4s?

**Solution**
The probability of getting a 4 on one die is $1/6$.

The probability of getting three 4s is: $P(4 \text{ and } 4 \text{ and } 4) = 1/6 \times 1/6 \times 1/6 = 1/216$.
When the outcome affects the second outcome, which is what we called dependent events.

8. What is the probability for you to choose two red cards in a deck of cards?

**Solution**
A deck of cards has 26 black and 26 red cards. The probability of choosing a red card randomly is: \( P(\text{red}) = \frac{26}{52} = \frac{1}{2} \)
The probability of choosing a second red card from the deck is now: \( P(\text{red}) = \frac{25}{51} \)
The probability: \( P(2\text{red}) = \frac{1}{2} \times \frac{25}{51} = \frac{25}{102} \)

**Rule of the Product**
The total probability of events that are independent is found out by multiplying the probability of the events. Let us see with the help of examples:

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**UNIT 6 SECTION 4 DEPENDENT EVENTS IN CONDITIONAL PROBABILITIES AND TREE DIAGRAMS**

**Introduction**
Dear Students,
In this unit 6 section 3, we learned the basic concepts of dependent and independent events and their applications to real life situations. In this section, we are going to learn dependent events using conditional probabilities and tree diagrams.
Tree and Venn Diagrams
Sometimes, when the probability problems are complex, it can be helpful to graph the situation. Tree diagrams and Venn diagrams are two tools that can be used to visualize and solve conditional probabilities.

A tree diagram is a special type of graph used to determine the outcomes of an experiment. It consists of ‘branches’ that are labelled with either frequencies or probabilities.

Tree diagrams can make some probability problems easier to visualize and solve. The following example illustrates how to use a tree diagram.

Example
1. In an urn, there are 11 balls. Three balls are red (R) and eight balls are blue (B). Draw two balls, one at a time, with replacement. With replacement means that you put the first ball back in the urn before you select the second ball. The tree diagram using frequencies that show all the possible outcomes follows.
Solution
On the diagram, the first set of branches represents the first draw. The second set of branches represents the second draw. Each of the outcomes is distinct.

In fact, we can list each red ball as $R_1$, $R_2$, and $R_3$ and each blue ball as $B_1$, $B_2$, $B_3$, $B_4$, $B_5$, $B_6$, $B_7$, and $B_8$.

Then the nine $RR$ outcomes can be written as: $R_1R_1; R_1R_2; R_1R_3; R_2R_1; R_2R_2; R_2R_3; R_3R_1; R_3R_2; R_3R_3$.

The other outcomes are similar.

There are a total of 11 balls in the urn. Draw two balls, one at a time, with replacement. There are $11(11) = 121$ outcomes, the size of the sample space.

a. List the 24 BR outcomes: $B_1R_1$, $B_1R_2$, $B_1R_3$, ...

Solution
a. $B_1R_1; B_1R_2; B_1R_3; B_2R_1; B_2R_2; B_2R_3; B_3R_1; B_3R_2; B_3R_3; B_4R_1; B_4R_2; B_4R_3; B_5R_1; B_5R_2; B_5R_3; B_6R_1; B_6R_2; B_6R_3; B_7R_1; B_7R_2; B_7R_3; B_8R_1; B_8R_2; B_8R_3$

b. Using the tree diagram, calculate $P(RR)$.

Solution

c. Using the tree diagram, calculate $P(RB \text{ OR } BR)$:

Solution

d. Using the tree diagram, calculate \( P(R \text{ on 1st draw AND } B \text{ on 2nd draw}) \).

**Solution**
d. \( P(R \text{ on 1st draw AND } B \text{ on 2nd draw}) = P(RB) = (3/11)(8/11) = 24/121 \)

e. Using the tree diagram, calculate \( P(R \text{ on 2nd draw GIVEN } B \text{ on 1st draw}) \).

**Solution**
e. \( P(R \text{ on 2nd draw GIVEN } B \text{ on 1st draw}) = P(R \text{ on 2nd}|B \text{ on 1st}) = 24/88 = 3/11 \)

This problem is a conditional one. The sample space has been reduced to those outcomes that already have a blue on the first draw. There are 24 + 64 = 88 possible outcomes (24 BR and 64 BB). Twenty-four of the 88 possible outcomes are BR. \( 24/88 = 3/11 \).

f. Using the tree diagram, calculate \( P(BB) \).

**Solution**
f. \( P(BB) = 64/121 \)

g. Using the tree diagram, calculate \( P(B \text{ on the 2nd draw given } R \text{ on the first draw}) \).

**Solution**
g. \( P(B \text{ on 2nd draw}|R \text{ on 1st draw}) = 8/11 \)

There are 9 + 24 outcomes that have \( R \) on the first draw (9 RR and 24 RB). The sample space is then 9 + 24 = 33.

24 of the 33 outcomes have \( B \) on the second draw. The probability is then 24/33 = 8/11.

2. Calculate the following probabilities using the tree diagram below:
a. \( P(RR) = \frac{3}{11} \cdot \frac{2}{10} = \frac{6}{110} = \frac{2}{55} \)

b. \( P(RB \text{ OR } BR) = \frac{3}{11} \cdot \frac{8}{10} + \frac{8}{11} \cdot \frac{3}{10} = \frac{48}{110} = \frac{24}{55} \)

c. \( P(R \text{ on 2nd}|B \text{ on 1st}) = \frac{3}{10} \)

d. \( P(R \text{ on 1st AND } B \text{ on 2nd}) = \frac{3}{11} \cdot \frac{8}{10} = \frac{24}{110} = \frac{12}{55} \)

e. \( P(BB) = \frac{8}{11} \cdot \frac{7}{10} = \frac{56}{110} = \frac{28}{55} \)

f. \( P(B \text{ on 2nd}|R \text{ on 1st}) = P(R|B) = \frac{8}{10} \)

**Summary**

In this section, we have learned how to use the tree diagrams to solve problems in probability. I hope we have all enjoyed the two problems we have solved above?

Thank you.

Let us proceed to section 5.

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**UNIT 6 SECTION 5 DEPENDENT EVENTS WITH CONDITIONAL PROBABILITY AND BAYES THEOREM**

**Introduction**

Dear Students,
In this unit 6 section 4, we learned the basic concepts of conditional probability using the tree diagrams and their applications to real life situations. In this section, we are going to extend the tree diagrams to Bayes theorem and solve practical problems. Say active and we all do the following activities!

**Conditional Probability**

Dear Students,

Suppose we know that a certain event ‘B’ has occurred. How does this impact the probability of some other ‘A’. This question is addressed by conditional probabilities. We write it as $P(A|B)$, the conditional probability of $A$ given $B$.

The conditional probability of $A$ given $B$ is expressed as $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

It is also useful to think of this formula $P(A \cap B) = P(A|B)P(B)$.

Thus, in Conditioning, $P(A) = P(A|B)P(B) + P(A|B)P(B)$.

More generally we can condition on a collection of $n$ events provided they are pairwise disjoint and add up to all the sample space as $P(A) = P(A|B_1)P(B_1) + P(A|B_2)P(B_2) + \cdots + P(A|B_n)P(B_n)$.

**Bayes’ Theorem**

Dear Students,

Bayes Theorem is a theorem in probability theory named for Thomas Bayes (1702-1761). In epidemiology, it is used to obtain the probability of disease in a group of people with some characteristic on the basis of the overall rate of that disease and of the likelihoods of that characteristic in healthy and diseased individuals.

The most familiar application is in clinical decision analysis where it is used for estimating the probability of a particular diagnosis given the appearance of some symptoms or test result.

Bayes’ theorem is a way to figure out conditional probability. Conditional probability is the probability of an event happening, given that it has some relationship to one or more other events. For example, your probability of getting a sitting place in the lecture room is connected to the time of the day you come to lectures, where you sit, and what conventions are going on at any time. Bayes’ theorem is slightly more nuanced. In a nutshell, it gives you the actual probability of an event given information about tests.

Mathematically, $P(A|B) = \frac{P(B|A)P(A)}{P(B)}$.

**Activities**
1. In a particular pain clinic, 10% of patients are prescribed narcotic pain killers. Overall, five percent of the clinic’s patients are addicted to narcotics (including pain killers and illegal substances). Out of all the people prescribed pain pills, 8% are addicts. If a patient is an addict, what is the probability that they will be prescribed pain pills?

**Solution**

Step 1: *Figure out what your event “A” is from the question.* That information is in the italicized part of this particular question. The event that happens first (A) is being prescribed pain pills. That’s given as 10%.

Step 2: *Figure out what your event “B” is from the question.* That information is also in the italicized part of this particular question. Event B is being an addict. That’s given as 5%.

Step 3: *Figure out the probability of event B (Step 2) given event A (Step 1).* In other words, find what (B|A) is. We want to know “Given that people are prescribed pain pills, what is the probability they are an addict?” That is given in the question as 8%, or .8.

Step 4: *Insert your answers from Steps 1, 2 and 3 into the formula and solve.*

\[
P(A|B) = \frac{P(B|A) \times P(A)}{P(B)} = \frac{0.08 \times 0.1}{0.05} = 0.16
\]

Therefore, the probability of an addict being prescribed pain pills is 0.16 or 16%.

2. A doctor is called to see a sick child. The doctor has prior information that 90% of sick children in that neighborhood have the flu, while the other 10% are sick with measles.

Let F stand for an event of a child being sick with flu and M stand for an event of a child being sick with measles. Assume for simplicity that \( F \cup M = \Omega \), i.e., there are no other maladies in that neighborhood. A well-known symptom of measles is a rash (the event of having which we denote R). Assume that the probability of having a rash if one has measles is \( P(R | M) = 0.95 \). However, occasionally children with flu also develop rash, and the probability of having a rash if one has flu is \( P(R | F) = 0.08 \). Upon examining the child, the doctor finds a rash.

What is the probability that the child has measles?

**Solution**

We use Bayes’s formula

\[
P(M | R) = \frac{P(R | M)P(M)}{P(R | M)P(M) + P(R | F)P(F)}
\]

This becomes

\[
P(M | R) = \frac{0.95 \times 0.10}{0.95 \times 0.10 + 0.08 \times 0.90} = 0.57
\]

3. Suppose we have 3 cards identical in form except that both sides of the first card are colored red, both sides of the second card are colored black, and one side of the third card is colored red and the other side is colored black. The 3 cards are mixed up in a hat, and 1 card is randomly selected and put down on the ground.

If the upper side of the chosen card is colored red, what is the probability that the other side is colored black?

**Solution**
Let RR, BB, and RB denote, respectively, the events that the chosen card is the red-red, the black-black, or the red-black card.

Letting R be the event that the upturned side of the chosen card is red, then we obtain,

\[
P(RB / R) = \frac{P(RR) \cap P(R)}{P(R)} = \frac{P(R / RB)P(RM)}{P(R / RR)P(RR) + P(R / RBP)P(RB) + P(R / BB)P(BB)}
\]

\[
P(RB / R) = \frac{(1/2)(1/3)}{(1)(1/3) + (1/2)(1/3) + (0)(1/3)} = \frac{1}{3}
\]

**Summary**

In this section, we have learned that conditional probability of A given B is expressed as

\[
P(A / B) = \frac{P(A \cap B)}{P(B)}.
\]

We have also learned that Bayes’ theorem gives you the actual probability of event ‘A’ given event ‘B’ as

\[
P(A / B) = \frac{P(B / A)P(A)}{P(B)}.
\]

Can you create your own problems and use the two theorems to solve them?

Congratulations!

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**UNIT 6 SECTION 6 APPLICATIONS OF PERMUATIONS AND COMBINATIONS IN PROBABILITY**

Dear Students,

In unit 6 section 1, we learned the counting principles. The two main ones were permutations and combinations.

What did we say about permutations?

What did we say about combinations?
Thank you for recalling these two important concepts in probability.

Again, in unit 6 section 5, we learned that conditional probability of A given B is

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \],

and the Bayes’ theorem of event ‘A’ given event ‘B’ as

\[ P(A \mid B) = \frac{P(B \mid A)P(A)}{P(B)} \].

In this section, we are going to learn how to solve problems in probability using permutations and combinations.

Take your pens, calculators and jotters and lets go!

Permutations and combinations

Dear Students,

Recall that the difference between combinations and permutations is ordering. With permutations we care about the order of the elements, whereas with combinations we do not order.
Also, recall that the permutation of $r$ objects from $n$ is given by $nPr = \frac{n!}{(n-r)!}$ and combination is $nCr = \frac{n!}{r!(n-r)!}$.

As applied to probability, $nPr = \frac{n!}{(n-r)!}$ and $nCr = \frac{n!}{r!(n-r)!}$ serve as the total or sample space while the selections serve as the numerators.

Dear Students,
Let us go straight to solving some problems and these concepts will be consolidated. Thank you.

**Activities**

1. A four-digit PIN is selected. What is the probability that there are no repeated digits?

**Solution**

There are 10 possible values for each digit of the PIN (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)

So there are $10 \times 10 \times 10 \times 10 = 10^4 = 10000$ total possible PINs.

To have no repeated digits, all four digits would have to be different, which is selecting without replacement. We could either compute $10 \times 9 \times 8 \times 7$, or notice that this is permutation $10P_4 = 5040$.

The probability of no repeated digits is the number of 4 digit PINs with no repeated digits divided by the total number of 4 digit PINs.

This probability is $\frac{10P_4}{10^4} = \frac{5040}{10000} = 0.504$

2. In a certain lottery company, 48 balls numbered 1 through 48 are placed in a machine and six of them are drawn at random. If the six numbers drawn match the numbers that a player had chosen, the player wins 1,000,000 Ghana cedis. In this lottery, the order the numbers are drawn in does not matter.

Compute the probability that you win the million-dollar prize if you purchase a single lottery ticket.

**Solution**
In order to compute the probability, we need to count the total number of ways six numbers can be drawn, and the number of ways the six numbers on the player’s ticket could match the six numbers drawn from the machine.

Since there is no stipulation that the numbers be in any particular order, the number of possible outcomes of the lottery drawing is \( \binom{48}{6} = 12,271,512 \).

Of these possible outcomes, only one would match all six numbers on the player’s ticket, so the probability of winning the grand prize is \( \frac{1}{12,271,512} \).

3. In the lottery company from question 2, if five of the six numbers drawn match the numbers that a player has chosen, the player wins a second prize of Ghc1,000. Compute the probability that she wins the second prize if she purchases a single lottery ticket.

**Solution**

The number of possible outcomes of the lottery drawing is \( \binom{48}{6} = 12,271,512 \).

In order to win the second prize, five of the six numbers on the ticket must match five of the six winning numbers.

In other words, we must have chosen five of the six winning numbers and one of the 42 losing numbers. The number of ways to choose 5 out of the 6 winning numbers is given by \( \binom{6}{5} = 6 \) and the number of ways to choose 1 out of the 42 losing numbers is given by \( \binom{42}{1} = 42 \).

Thus the number of favourable outcomes is then given by \( \binom{6}{5} \times \binom{42}{1} = 6 \times 42 = 252 \).

So the probability of winning the second prize is \( \frac{\binom{6}{5} \binom{42}{1}}{\binom{48}{6}} = \frac{252}{12,271,512} = 0.0000205 \).

4. Compute the probability of randomly drawing five cards from a deck and getting exactly one Ace.

**Solution**

In many card games (such as poker) the order in which the cards are drawn is not important (since the player may rearrange the cards in his hand any way he chooses). So, we will assume that this
is the case. Thus we use combinations to compute the possible number of 5-card hands, \( \binom{52}{5} \).

This number will go in the denominator of our probability formula, since it is the number of possible outcomes.

For the numerator, we need the number of ways to draw one Ace and four other cards (none of them Aces) from the deck.

Since there are four Aces and we want exactly one of them, there will be \( \binom{4}{1} \) ways to select one Ace.

Since there are 48 non-Aces and we want 4 of them, there will be \( \binom{48}{4} \) ways to select the four non-Aces.

Now we use the Basic Counting Rule to calculate \( \binom{4}{1} \times \binom{48}{4} \) ways to choose one ace and four non-Aces.

Putting this all together, we have

\[
P(\text{one ace}) = \frac{\binom{4}{1} \binom{48}{4}}{\binom{52}{5}} = \frac{778320}{2598960} = 0.299
\]

**Summary**

Dear Students,

You have noticed how Permutation and Combination is a very important topic in this unit. Through permutations and combinations, we counted various arrangements in unit 6 section 1 that can be made from a certain group. In this unit 6 section 6, we have used the concepts to solve concepts with a diverse forms and structure.

Continue to explore more applications of Permutations and Combinations in your everyday life.

Congratulations!