TEACHING AND ASSESSING JUNIOR HIGH SCHOOL MATHEMATICS (INTEMEDIARY)

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

UNIT 1
MEASUREMENT, SHAPE AND SPACE

INFORMAL GEOMETRY AND SPATIAL SENSE

Spatial sense is an intuitive feel for shape and space. It involves the concepts of traditional geometry, including an ability to recognize, visualize, represent, and transform geometric shapes. It also involves other, less formal ways of looking at two- and three-dimensional space, such as paper-folding, transformations, tessellations, and projections. Geometry is all around us in art, nature, and the things we make. Students of geometry can apply their spatial sense and knowledge of the properties of shapes and space to the real world.

Meaning and Importance

Geometry is the study of spatial relationships. It is connected to every strand in the mathematics curriculum and to a multitude of situations in real life. Geometric figures and relationships have played an important role in society's sense of what is aesthetically pleasing. From the Greek discovery and architectural use of the golden ratio to M. C. Escher's use of tessellations to produce some of the world's most recognizable works of art, geometry and the visual arts have had strong connections. Well-constructed diagrams allow us to apply knowledge of geometry, geometric reasoning, and intuition to arithmetic and algebra problems. The use of a rectangular array to model the multiplication of two quantities, for instance, has long been known as an effective strategy to aid in the visualization of the operation of multiplication. Other mathematical concepts which run very deeply through modern mathematics and technology, such as symmetry, are most easily introduced in a geometric context. Whether one is designing an electronic circuit board, a building, a dress, an airport, a bookshelf, or a newspaper page, an understanding of geometric principles is required.

DEVELOPMENT AND EMPHASES

Traditionally, elementary school geometry instruction has focused on the categorization of shapes; at the secondary level, it has been taught as the prime example of a formal deductive system. While these perspectives of the content are important, they are also limiting. In order to develop spatial sense, students should be exposed to a broader range of geometric activities at all grade levels.

By virtue of living in a three-dimensional world, having dealt with space for five years, children enter school with a remarkable amount of intuitive geometric knowledge. The geometry curriculum should take advantage of this intuition while expanding and formalizing the students' knowledge. In early elementary school, a rich, qualitative, hands-on study of geometric objects helps young children develop spatial sense and a strong intuitive grasp of geometric properties and relationships. Eventually they develop a comfortable vocabulary of appropriate geometric terminology. In the middle school years, students should begin to use their knowledge in a more analytical manner to solve problems, make conjectures, and look for patterns and generalizations. Gradually they develop the ability to make inferences and logical deductions based on geometric relationships and to use spatial intuition to develop more generic mathematical problem-solving skills. In high school, the study of geometry expands to address coordinate, vector, and transformational viewpoints which utilize both inductive and deductive reasoning. Geometry instruction at the high school level should not be limited to formal deductive proof and simple measurement activities, but should include the study of geometric transformations, analytic geometry, topology, and connections of geometry with algebra and other areas of mathematics.

At all grade levels, the study of geometry should make abundant use of experiences that require active student involvement. Constructing models, folding paper cutouts, using mirrors, pattern blocks, geoboards, and tangrams, and creating geometric computer graphics all provide opportunities for students to learn by doing, to reflect upon their actions, and to communicate their observations and conclusions. These activities and others of the same type should be used to achieve the goals in the seven specific areas of study that constitute this standard and which are described below.

In their study of **spatial relationships**, young students should make regular use of concrete materials in hands-on activities designed to develop their understanding of objects in space. The early focus should be the description of the location and orientation of objects in relation to other objects. Additionally, students can begin an exploration of symmetry, congruence, and similarity. Older students should study the two dimensional representations of three-dimensional objects by sketching shadows, projections, and perspectives.

In the study of **properties of geometric figures**, students deal explicitly with the identification and classification of standard geometric objects by the number of edges and vertices, the number and shapes of the faces, the acuteness of the angles, and so on. Cut-and-paste constructions of paper models, combining shapes to form new shapes and decomposing complex shapes into simpler ones are useful exercises to aid in exploring shapes and their properties. As their studies continue, older students should be able to understand and perform classic constructions with straight edges and compasses as well as with appropriate computer software. Formulating good mathematical definitions for geometric shapes should eventually lead to the ability to make hypotheses concerning relationships and to use deductive arguments to show that the relationships exist.

The standard **geometric transformations** include translation, rotation, reflection, and scaling. They are central to the study of geometry and its applications in that these movements offer the most natural approach to understanding congruence, similarity, symmetry, and other geometric relationships. Younger children should have a great deal of experience with *flips*, *slides*, and *turns* of concrete objects, figures made on geoboards, and drawn figures. Older students should be able to use more formal terminology and procedures for determining the results of the standard transformations. An added benefit of experience gained with simple and composite transformations is the mathematical connection that older students can make to functions and function composition.

Coordinate geometry provides an important connection between geometry and algebra. Students can work informally with coordinates in the primary grades by finding locations in the room, and by studying simple maps of the school and neighborhood. In later elementary grades, they can learn to plot figures on a coordinate plane, and still later, study the effects of various transformations on the coordinates of the points of two- and three-dimensional figures. High-school students should be able to represent geometric transformations algebraically and interpret algebraic equations geometrically.

Measurement and geometry are interrelated, and an understanding of the **geometry of measurement** is necessary for the understanding of measurement. In elementary school, students should learn the meaning of such geometric measures as length, area, volume and angle measure and should be actively involved in the measurement of those attributes for all kinds of two- and three-dimensional objects, not simply the standard ones. Throughout school, they should use measurement activities to reinforce their understanding of geometric properties. All students should use these experiences to help them understand such principles as the quadratic change in area and cubic change in volume that occurs with a linear change of scale.

Trigonometry and its use in making indirect measurements provides students with another view of the interrelationships between geometry and measurement.

Geometric modeling is a powerful problem-solving skill and should be used frequently by both teachers and students. A simple diagram, such as a pie-shaped graph, a force diagram in physics, or a dot-and-line illustration of a network, can illuminate the essence of a problem and allow geometric intuition to aid in the approach to a solution. Visualization skills and understanding of concepts will both improve as students are encouraged to make such models.

The relationship between geometry and **deductive reasoning** originated with the ancient Greek philosophers, and remains an important part of the study of geometry. A key ingredient of deductive reasoning is being able to recognize which statements have been justified and which have been assumed without proof. This is an ability which all students should develop in all areas, not just geometry, or even just mathematics! At first, deductive reasoning is informal, with students inferring new properties or relationships from those already established, without detailed explanations at every step. Later, deduction becomes more formal as students learn to use all permissible assumptions in a proof and as all statements are systematically justified from what has been assumed or proved before. The idea of deductive proof should not be confused with the specific two-column format of proof found in most geometry textbooks. The reason for studying deductive proof is to develop reasoning skills, not to write out arguments in a particular arrangement. Note that proof by mathematical induction is another deductive method that should not be neglected.

Much of the current thinking about the development of geometric thinking in students comes from the work of a pair of Dutch researchers, Pierre van Hiele and Dina van Hiele-Geldof. Their model of geometric thinking identifies five levels of development through which students pass when assisted by appropriate instruction.

- Visual recognition of shapes by their appearances as a whole (level 0)

- Analysis and description of shapes in terms of their properties (level 1)
- Higher "theoretical" levels involving informal deduction (level 2)
- Formal deduction involving axioms and theorems (level 3)
- Work with abstract geometric systems (level 4).

(Geddes & Fortunato, 1993)

Although the levels are not completely separate and the transitions are complex, the model is very useful for characterizing levels of students' thinking. Consistently, the research shows that appropriately targeted instruction is critical to children's movement through these levels. Stagnation at early levels is the frequent result of a geometry curriculum that never moves beyond identification of shapes and their properties. The discussion in this K-12 Overview draws on this van-Hiele model of geometric thinking.

IN SUMMARY, students of all ages should recognize and be aware of the presence of geometry in nature, in art, and in human-built structures. They should realize that geometry and geometric applications are all around them and, through study of those applications, come to better understand and appreciate the role of geometry in life. Carpenters use triangles for structural support, scientists use geometric models of molecules to provide clues to understanding their chemical and physical properties, and merchants use traffic-flow diagrams to plan the placement of their stock and special displays. These and many, many more examples should leave no doubt in students' minds as to the importance of the study of geometry.

NOTE: *Although each content standard is discussed in a separate chapter, it is not the intention that each be treated separately in the classroom. Indeed, as noted in the Introduction to this Framework, an effective curriculum is one that successfully integrates these areas to present students with rich and meaningful cross-strand experiences.*

THE DEVELOPMENT OF GEOMETRIC THINKING

The van Hiele Levels of Geometric Thought

There is some well-established research that has been influencing school curriculum development internationally for many years now, but the practical details are still unknown to most teachers. This research began in the 1950's with a husband and wife team in the Netherlands, Pierre and Dina van Hiele. Pierre van Hiele has continued to develop the theory over the years, and many other researchers around the world have investigated its basis and application in various ways. The main theory emphasises that despite some natural development of spatial thinking, deliberate instruction is needed to move children through several levels of geometric understanding and reasoning skill. It is based on the firm belief that it is inappropriate to teach children Euclidean geometry following the same logical construction of axioms, definitions, theorems and proofs that Euclid used to construct the system. Children don't think on a formal deductive level, and therefore can only memorise geometric facts and 'rules', but not understand the relationships between the ideas, if taught using this approach.

The van Hiele theory puts forward a hierarchy of levels of thinking spanning the ages of about five years through to academic adults. Originally there were five levels, that have been adapted and renamed by various researchers, but now van Hiele concentrates on the three levels that cover the normal period of schooling. The main content focus is on two-dimensional (plane) shape.

Level 1: Visual

This level begins with 'nonverbal thinking'. Shapes are judged by their appearance and generally viewed as 'a whole', rather than by distinguishing parts. Although children begin using basic shape names, they usually offer no explanation or associate the shapes with familiar objects. For example, a child might say, "It's a square because it looks like one", or "I know it's a rectangle because it looks like a box". This could be likened to young children's ability to recognise some words by sight, before they understand the individual letter sounds and how they blend together to form words.

Level 2: Descriptive

At this level, children can identify and describe the component parts and properties of shapes. For example, an equilateral triangle can be distinguished from other triangles because of its three equal sides, equal angles and symmetries. Children need to develop appropriate language to go with the new specific concepts. However, at this stage the properties are not 'logically ordered', which means that the children do not perceive the essential relationships between the properties. So, with the equilateral triangle for example, they do not understand that if a triangle has three equal sides it must have three equal angles.

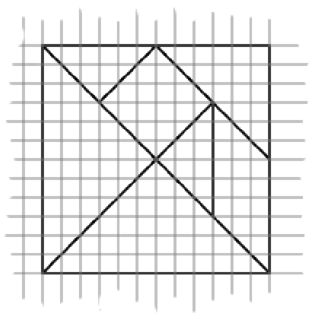
Level 3: Informal Deduction

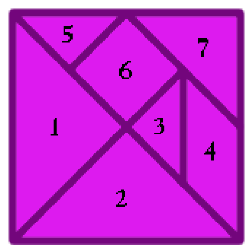
In this level, the properties of shapes are logically ordered. Students are able to see that one property precedes or follows from another, and can therefore deduce one property from another. They are able to apply what they already know to explain the relationships between shapes, and to formulate definitions. For example, they could explain why all squares are rectangles. Although informal deduction such as this forms the basis of formal deduction, the role of axioms, definitions, theorems and their converses, is not understood.

ACTIVITY PHASES TO DEVELOP GEOMETRIC THINKING

Included in van Hiele's theory is a sequence of five phases of activity types that are designed to promote the movement of children's thinking from one level to the next. There are a variety of ways to use these phases when planning lessons. For example, a teacher might choose to deal with the topic of squares, and so design all the tasks to involve the manipulation, construction, discussion and so on, of squares. Another way to use the phases is to take a more general approach and explore a number of different shapes simultaneously. Cycling through these phases using materials such as tangrams and various mosaic puzzles enables children to build a rich background in visual and descriptive thinking about shapes and their properties.

Activity Examples: Traditional seven-piece tangram sets are commonly available and so have been used as the basis of activities to illustrate each phase. The topic is not 'Tangrams', but the 'Properties of 2D Shapes'. The activities are not really age specific, but are probably best suited to children in the range of 7 to 10 years old. It is desirable for each child to have his or her own tangram, so multiple copies could be made from card and stored in envelopes. The pieces in the example below have been numbered for easy reference in this article, but it is also useful to do this (or colour-code) with the children's sets. (Find or make a tangram set and use it to try out the activities as you read. Instructions for folding a tangram can be found here or one can easily be drawn using grid paper as shown below).





Phase 1: Inquiry

Learning begins with play! The children should be encouraged to freely explore the materials and hence discover some properties and structures. While the children are playing, the teacher has the opportunity to observe and informally assess the children's thinking and language.

Example: Give each child a tangram set and simply ask 'What can you do with these pieces?' Encourage the children to share and talk about the shapes and pictures they have made. Allow plenty of time for children to freely explore the pieces. During this play the children will become familiar with the size and shape of the pieces, and they begin to see how they fit together. In other words, they begin to discover the properties and relationships.

Phase 2: Direct Orientation

Activities are presented in such a way that children's attention is focussed on particular characteristics of the shapes or puzzle pieces. Ideas for directed activities will come from watching the children at play, as well some pre-planned tasks.

Example: In free play a child may have used pieces 3 and 5 to make piece 6. So ask the children to find out whether all the pieces can be made from two smaller pieces. They might do this by fitting the two pieces together on top of the larger piece. Other questions to explore are:

- Which pieces can be made from three other pieces? Have the children record what they find by tracing around the shapes.
- One activity will often lead to another; How many different ways can the largest triangle (1) be made using the other pieces?
- Choose two shapes. How many different shapes can you make with them? Draw them all and give them names. Use three shapes to make a new shape (not like one of the other pieces). Trace around it. How many ways can you make this shape? Can you make it with and without the numbers showing? Draw all the solutions.

From these activities the children gain a more specific understanding of the properties of the shapes. They will notice the lengths of sides, some that are equal and others that are half or double the length of others. They will visualise how the angles will fit together and how to flip and turn the shapes to fit various spaces.

Phase 3: Explication

This involves tasks and games that deliberately develop the vocabulary associated with the ideas that have been encountered so far, while continuing to explore the properties of the shapes. The teacher clarifies terms the children are already using and introduces new terms. Activities that encourage the children to use the vocabulary as they talk and write about their experiences should be utilised.

Example: During some workshops, the teacher names the shapes; square, isosceles triangle, and parallelogram, as well as other vocabulary associated with the properties; similar, equal sides, equal angles, right angle, symmetry and parallel sides. Questions like the following will provide opportunities for the terms to use as well as clarifying the associated concepts:

- Which shapes have a right angle? Match them up in a pile. (What size are the other angles?)
- Trace around each shape on paper and cut it out. How many lines of symmetry does each shape have?
- What is the same about all the triangles?
- Which shapes have parallel sides?
- Which shapes have sides the same length within themselves? Which shapes have sides the same length as other shapes? half/double the length?
- Play 'Mystery Bag' where a child feels a hidden shape and describes it to the rest of the class, who try to name the shape.

The children should be encouraged to explain and demonstrate what they find out, perhaps using pieces on an overhead projector, or making a display poster.

Phase 4: Free Orientation

The children engage in activities and problem-solving tasks that are open-ended or can be completed in different ways. The aim is for children to utilise what they have learned and become more proficient.

Example: The children should engage in more challenging tasks, that draw on the knowledge and skills previously developed; such as:

- How many ways can you make a square from some or all of the pieces?
- Complete classic tangram puzzles of outlines of birds and animals.
- Draw a completed tangram square (like the one illustrated above) on an 8x8 square grid, examine the pieces carefully in relation to the grid, then work out a way to enlarge all the pieces.

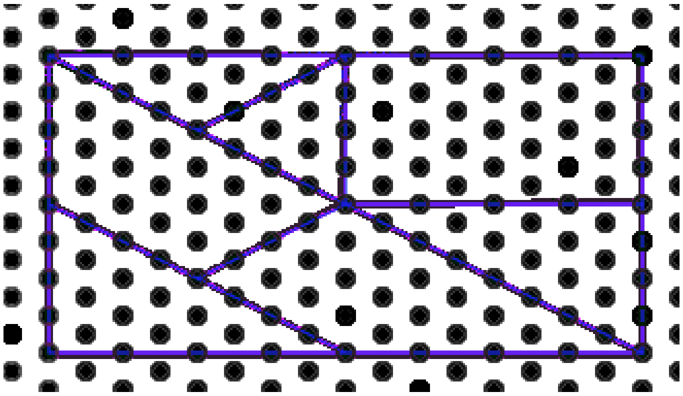
Phase 5: Integration

Opportunities are given for the children to pull together their new knowledge and reflect on it as a whole. They should be able to express or summarise what they have learned in some way.

Example : Small groups of children can design charts, class books and displays that present what they have learned about the tangram shapes. Pairs of children could prepare "What shape am I?" cards which could be used by individuals or in team games. Class games like "Twenty Questions" will help children to logically organise their knowledge of properties of the shapes. Individual assessment might be carried out by giving children open-ended tasks, like 'Choose a shape and write everything you can about it', and 'Choose another shape and design a What shape am I? Card'.

Outcomes of the Five Phases of Activities

As a consequence of engaging in a series of activities such as those described above, the majority of children will have progressed from simply recognising some shapes, to being able to discuss the shapes in terms of specific geometric properties and perhaps make some comparisons between shapes. In other words, they will have moved from Level 1 Visual thinking to Level 2 Descriptive thinking in regards to the shapes treated in the activities. Although the knowledge and vocabulary developed in respect to this specific set of shapes will assist in the study of different shapes, many more similar experiences will be needed to develop geometric thinking about different shapes. For example, the puzzle shapes presented below would provide good opportunities for the children to explore types of triangles and quadrilaterals. (Note that it is easily copied using equilateral-triangle dot or grid paper).



EXPLORING SHAPES AND THEIR PROPERTIES

In geometry, **2d shapes** and 3d shapes are explained widely to make you understand the different types of objects you come across in real life. These shapes have their own pattern and properties. Depending on many factors, such as angle, sides, length, height, width, area, volume, etc., the shapes can vary. These 2D and 3D shapes have been taught to us since our primary classes. Let us learn various types of two-dimensional shapes here, in this article.

2D SHAPES

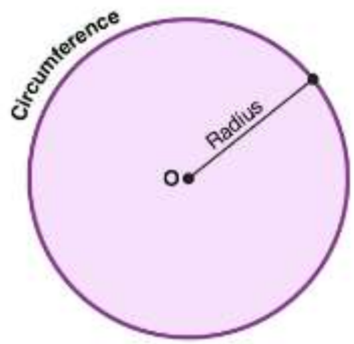
In maths, 2d shapes can be defined as the plane figures that can be drawn on a flat (or plane) surface or a piece of paper. All the 2d shapes have various parameters such as area and perimeter. Some of the 2d shapes contain sides and corners, whereas some have curved boundaries.

2D Shapes Names: Circle, Triangle, Square, Rectangle, Pentagon, Octagon.

The basic types of 2d shapes are a circle, triangle, square, rectangle, pentagon, quadrilateral, hexagon, octagon, etc. Apart from the circle, all the shapes are considered as polygons, which have sides. A polygon which has all the sides and angles as equal is called a regular polygon. Including the circle, an ellipse is also a non-polygon shape. Both circle and ellipse have a curved shape, whereas the polygons have a closed structure with sides. Now let us discuss some shapes one by one.

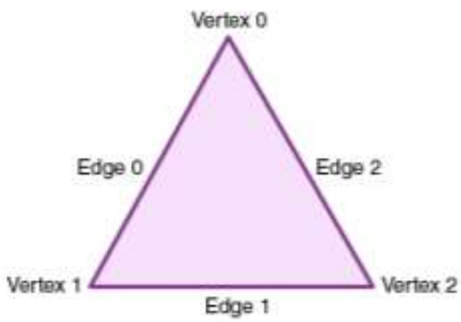
Circle

A circle is a closed 2d figure in which the set of all the points in the plane is equidistant from a given point called "center". The distance from the center to the outer line of the circle is called a radius. The example of the circle in real life are wheels, pizzas, orbit, etc.



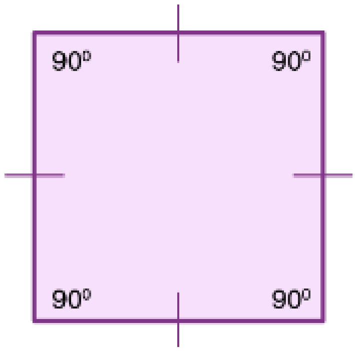
Triangle

A triangle is a three-sided polygon (2d Shape) which has three edges and three vertices. The sum of all the three angles of a triangle is equal to 180°. Pyramids are the best example of a triangle shape. You can also learn the properties of triangle here.



Square

A square is a four-sided polygon (2d Shape), whose four sides are equal in length and all the angles are equal to 90°. It is considered to be a two-dimensional regular quadrilateral. The diagonals of the square also bisect each other at 90°. A wall or a table where all the sides are equal are the examples of square shape.

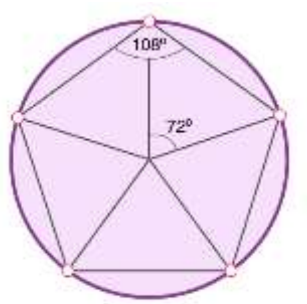


Rectangle

A rectangle is a 2d shape which has four sides, where the opposite sides are equal and parallel to each other. All the angles of a rectangle are equal to 90°. A brick, TV, cardboard, which has length and breadth are examples of the rectangle.

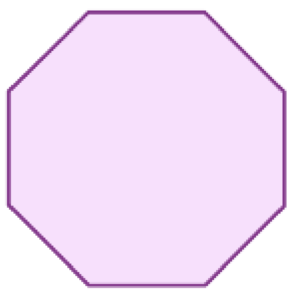
Pentagon

A pentagon is a five-sided polygon (2d Shape), and it can be regular or irregular. In the case of a regular pentagon, each interior angle is equal to 108°, and each exterior angle measures 72°. It has five diagonals. The Pentagon building, which is the headquarters of the US Department of Defense, is a great example of the pentagon shape.



Octagon

An octagon is an eight-sided polygon which can be either regular or irregular. It is a 2d shape which has eight angles. The sum of all the interior angles of an octagon is 1080°. The stop sign board has an octagon shape, which you can see on the roadside.



2D Shapes Properties

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Go through the below to learn all the properties of 2D shapes.

2 D Shapes	Properties of 2 D Shapes		
Square	Four equal sides	Four equal angles(90°)	Four axes of symmetry
Rectangle	2 sets of 2 equal sides	Four equal angles(90°)	Two axes of symmetry
Triangle	It can have no, 2 or 3 equal sides	It can have no, 2 or 3 equal angles	It can have up to 2 axes of symmetry
Circle	Constant diameter and radius	The total angle of a circle is equal to 360 degrees	Almost infinite axes of symmetry going through the centre
pentagon	5 sides (can be equal or unequal)	5 angles (can be equal or unequal)	It can have up to 5 axes of symmetry
hexagon	6 sides (can be equal or unequal)	6 angles (can be equal or unequal)	It can have up to 6 axes of symmetry
Octagon	8 sides (can be equal or unequal)	8 angles (can be equal or unequal)	It can have up to 8 axes of symmetry
Parallelogram	2 sets of 2 equal sides	2 sets of 2 equal angles	Usually no axes of symmetry
Rhombus	All sides the same length	2 sets of 2 equal angles	2 lines of symmetry
Trapezium	At least 2 parallel sides	Can have pairs of equal angles	It can have a line of symmetry

Area and Perimeter of 2D Shapes

The area is the region covered by a 2d shape on a plane. The areas for different shapes are given below:

2d Shape	Area	Perimeter
Circle	Πr^2 (R is the radius of the circle)	$2\pi r$
Triangle	½ (Base x height)	Sum of three sides
Square	$Side^2$	4(Side)
Rectangle	Length x Breadth	2(Length + Breadth)
Rhombus	½ (Product of diagonals)	4(Side)
Parallelogram	Base x Height	2 (Base + Side)

2d Shapes and 3d Shapes

We know that 2d shapes are flat figures and 3d shapes are solid figures. Below are the few comparisons of these two types of shapes.

2d Shapes	3d Shapes
It is a shape surrounded by three or more straight lines in a plane and sometimes with a closed curve.	If a shape is surrounded by a no. of surfaces or planes then it is a 3D shape.
These shapes have no depth or height.	These are also called solid shapes and unlike 2D they have height or depth.
These shapes have only two dimensions, say length and breadth, whereas curved shapes such as circle and ellipse have radii.	These are shapes containing three dimensions such as depth (or height), breadth and length.
Area, perimeter can be found for these shapes.	We can calculate their volume, CSA, LSA or TSA.
Examples: Circle, Triangle, Quadrilaterals, Polygons, etc.	Examples: Cube, Cuboid, Sphere, Cylinder, Cone, etc.

3D SHAPES

In Geometry, 3D shapes are known as three-dimensional shapes or solids. 3D shapes have three different measures such as length, width, and height as its dimensions. The only difference between 2D shape and 3D shapes is that 2D shapes do not have a thickness or depth.

Usually, 3D shapes are obtained from the rotation of the 2D shapes. The faces of the solid shapes are the 2D shapes. Some examples of the 3D shapes are a cube, cuboid, cone, cylinder, sphere, prism and so on.

TYPES OF 3D SHAPES

The 3D shapes consist of both curved shaped solid and the straight-sided polygon called the polyhedron. The polyhedrons are also called the polyhedra, which are based on the 2D shapes with straight sides. Now, let us discuss the details about the polyhedrons and curved solids.

Polyhedrons

Polyhedrons are 3D shapes. As discussed earlier, polyhedra are straight-sided solids, which has the following properties:

- Polyhedrons should have straight edges.
- It should have flat sides are called the faces
- It must have the corners, called vertices

Like polygons in two-dimensional shapes, polyhedrons are also classified into regular and irregular polyhedrons and convex and concave polyhedrons.

The most common examples of polyhedra are:

- Cube: It has 6 square faces, 8 vertices and 12 edges
- Cuboid: It has 6 rectangular faces, 8 vertices and 12 edges
- Pyramid: It has a polygon base, straight edges, flat faces and one vertex
- Prism: It has identical polygon ends and flat parallelogram sides

Some other examples of regular polyhedrons are tetrahedrons, octahedrons, dodecahedrons, icosahedrons, and so on. These regular polyhedrons are also known as platonic solids, whose faces are identical to each face.

For example, the most commonly used example of a polyhedron is a cube, which has 6 faces, 8 vertices, and 12 edges.

Curved Solids

The 3D shapes that have curved surfaces are called curved solids. The examples of curved solids are:

- Sphere: It is a round shape, having all the points on the surface equidistant from center
- Cone: It has a circular base and a single vertex
- Cylinder: It has parallel circular bases, connected through curved surface

Faces Edges and Vertices

Faces, edges and vertices are three important measures of 3D shapes, that defines their properties.

- Faces – A face is a curve or flat surface on the 3D shapes
- Edges – An edge is a line segment between the faces
- Vertices – A vertex is a point where the two edges meet

PROPERTIES OF 3D SHAPES

As we already discussed above the properties of 3D shapes are based on their faces, edges and vertices. Thus, we can have a brief of all the properties here in the table.

Cube	• 6 square faces • 8 vertices • 12 edges
Cuboid	• 6 rectangular faces • 8 vertices • 12 edges

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Cone	• 2 faces (circular base and curved surface) • 1 vertex • 1 edge
Cylinder	• 3 faces • 2 edges • 0 vertices
Sphere	• 1 curved surface • 0 edges • 0 vertices
Tetrahedron	• 4 faces • 6 edges • 4 vertices
Triangular prism	• 5 faces • 9 edges • 6 vertices
Square-based pyramid	• 5 faces • 8 edges • 5 vertices

SURFACE AREA AND VOLUME OF 3D SHAPES

The two different measures used for measuring the 3D shapes are:

- Surface Area
- Volume

Surface Area is defined as the total area of the surface of the two-dimensional object. The surface area is measured in terms of square units, and it is denoted as "SA". The surface area can be classified into three different types. They are:

- Curved Surface Area (CSA) – Area of all the curved regions
- Lateral Surface Area (LSA) – Area of all the curved regions and all the flat surfaces excluding base areas
- Total Surface Area (TSA) – Area of all the surfaces including the base of a 3D object

Volume is defined as the total space occupied by the three-dimensional shape or solid. It is measured in terms of cubic units and it is denoted by "V".

3D SHAPES FORMULAS

The formulas of 3D shapes related to surface areas and volumes are:

Name of the Shapes	Formulas
Cube	• TSA = $6a^2$ (square units) • LSA = $4a^2$ (square units) • Volume = a^3 (cubic units)
Cuboid	• TSA = 2 (lw + wh + lh) (square units)

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

	• LSA = 2h(l + w) (square units) • Volume = a^3 (cubic units)
Cone	• TSA = πr(l + r) (square units) • CSA = πrl (square units) • Volume = (1/3)πr^2h (cubic units)
Cylinder	• TSA = 2πr(h+r) (square units) • Volume = πr^2h (cubic units)
Sphere	• SA = 4πr^2 square units • Volume = (4/3)πr^3 cubic units

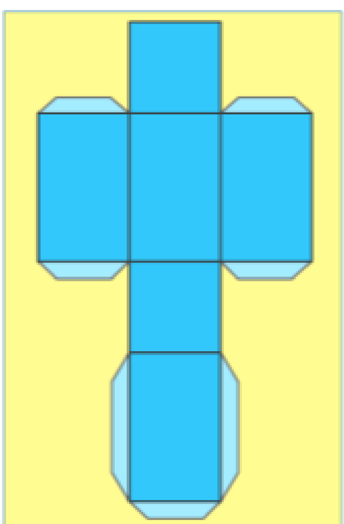
EXPLORING NETS OF 3D SHAPES

A net is a flattened out three-dimensional solid. It is the basic skeleton outline in two dimensions, which can be folded and glued together to obtain the 3D structure. Nets are used for making 3D shapes. Let us have a look at nets for different solids and its surface area and volume formula.

Cuboid

A cuboid is also known as a rectangular prism. The faces of the cuboid are rectangular. All the angle measures are 90 degrees.

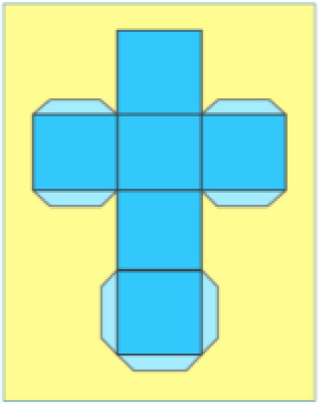
Take a matchbox. Cut along the edges and flatten out the box. This is the net for the cuboid. Now if you fold it back and glue it together similarly as you opened it, you get the cuboid.



Cube

A cube is defined as a three-dimensional square with 6 equal sides. All the faces of the cube have equal dimension.

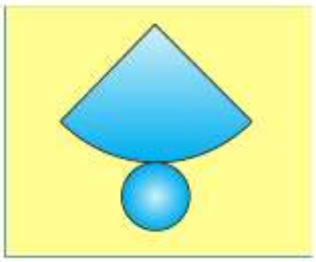
Take a cheese cube box and cut it out along the edges to make the net for a cube.



Cone

A cone is a solid object that has a circular base and has a single vertex. It is a geometrical shape that tapers smoothly from the circular flat base to a point called the apex.

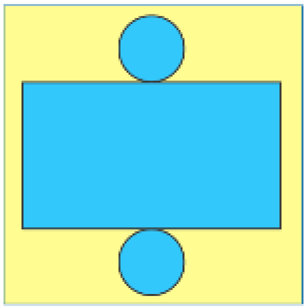
Take a birthday cap which is conical. When you cut a slit along its slant surface, you get a net for cone.



Cylinder

A cylinder is a solid geometrical figure, that has two parallel circular bases connected by a curved surface.

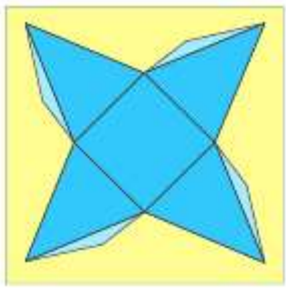
When you cut along the curved surface of any cylindrical jar, you get a net for the cylinder. The net consists of two circles for the base and the top and a rectangle for the curved surface.



Pyramid

A pyramid, also known as a polyhedron. A pyramid can be any polygon, such as a square, triangle and so on. It has three or more triangular faces that connect at a common point is called the apex.

The net for a pyramid with a square base consists of a square with triangles along its four edges.



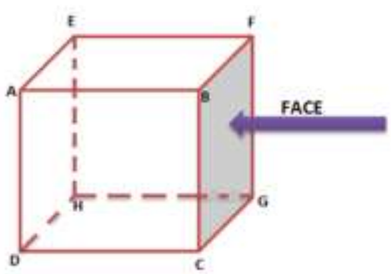
HAND SKETCHING OF COMMON SOLIDS

EXPLORING RELATIONSHIP AMONG FACES, EDGES AND VERTICES

A 3D shape or an object is made up of a combination of certain parts. Most of the solid figures consist of polygonal regions. These regions are- faces, edges, and vertices. Solid geometric shapes which have faces, edges and vertices are known as polyhedrons.

Faces of 3D Shapes

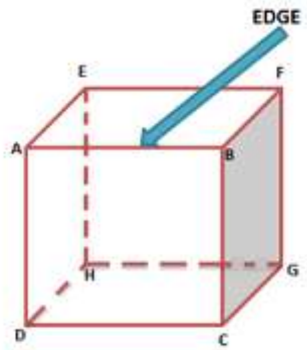
- The flat surface of a polyhedron is its face. Solid shapes can have more than one face. The cube shown below has 6 faces viz. ABCD, EFGH, ADHE, DHGC, BFGC, and AEFB.



- Cubes and cuboids have 6 faces.
- Cones have a flat face and a curved face.
- Cylinders have 2 flat faces and a curved face.
- A sphere has a curved face.

Edges of 3D Shapes

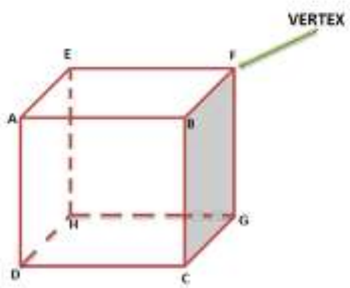
- The faces meet each other at edges. Edges are straight lines which serve as the junction of two faces. The cube shown below has 12 edges namely AB, BF, EF, AE, AD, DH, EH, HG, FG, BC, CG, and CD.



- Cubes and cuboids have 12 edges.
- Cones have 1 edge.
- Cylinders have 2 edges.
- A sphere has no edge.

Vertices of 3D Shapes

- The points of intersection of edges denote the vertices. Vertices are represented by points. In the cube shown below A, B, C, D, E, F, G, and H are the 8 vertices of the cube.

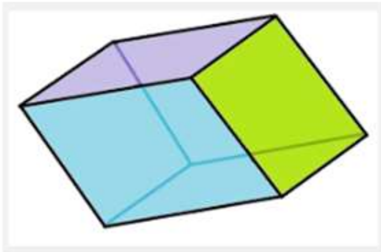


- Cubes and cuboids have 8 vertices.
- Cones have 1 vertex.
- Cylinders have no vertex.
- Spheres have no vertex (the surface is a curve).

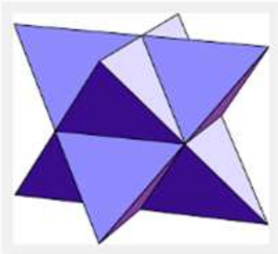
Now that, we are familiar with polyhedrons let's move onto to their types.

TYPES OF POLYHEDRON

- **Convex Polyhedron:** If the surface of a polyhedron (which consists of its faces, edges, and vertices) does not intersect itself and the line segment connecting any two points of the polyhedron lies within its interior part or surface then such a polyhedron is a convex polyhedron.



- **Concave Polyhedron:** A non-convex polyhedron is termed as a concave polyhedron.



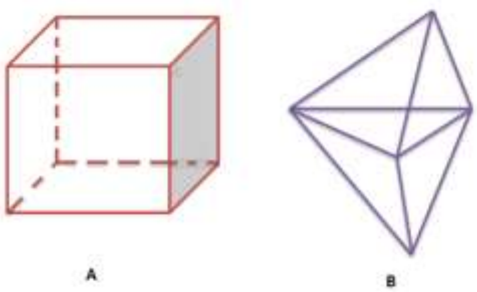
Euler's Formula

According to Euler's formula for any convex polyhedron, the number of Faces (F) and vertices (V) added together is exactly two more than the number of edges (E).

F + V = 2 + E

Polyhedron	Faces	Vertices	Faces + Vertices	Edges	Edges+2
Tetrahedron	4	4	8	6	8
Cube	6	8	14	12	14
Octahedron	8	6	14	12	14
Icosahedron	20	12	32	30	32
Dodecahedron	12	20	32	30	32

A polyhedron is known as a regular polyhedron if all its faces constitute regular polygons and at each vertex the same number of faces intersect.

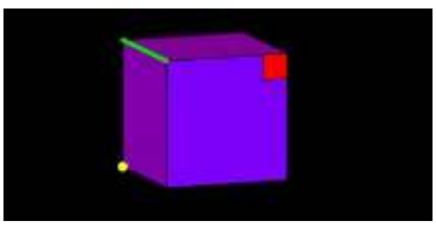


CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Fig A is a regular polyhedron as all the faces are regular polygons and B is an irregular polygon.

It is said that in 1750, Euler derived the well known formula V + F – E = 2 to describe polyhedrons. At first glance, Euler's formula seems fairly trivial. Edges, faces and vertices are considered by most people to be the characteristic elements of polyhedron. Surprisingly however, concise labelling of such characteristics was not presented until the 1700's. Leonhard Euler, recognizing the deficiency, began his investigation of general polyhedra, and the relationship between their elements.

Euler emphasized five major components of a polyhedron in an attempt to find the relationship between them. These five components were vertices, (a location where 2 or more edges meet), faces (contained and defined by 3 or more edges), edges (defined as the "ridge or sharp edge"[2] of a polyhedron), sides (used to refer to the sides of each face), and plane angles (the angle found at a vertex, contained by 2 sides). These definitions, contrasted with the features that Euclid had relied on previously, right angles, and bases, led to far more possible relationships between characteristics.



From above, green denotes the edge of the cube; red denotes the plane angle;

the yellow sphere occurs at a vertex; and all faces are shaded purple.

Sides are in black, they surround the purple faces.

In the remainder, let: - V be the number of vertices,

- F be the number of faces,
- E be the number of edges,
- S be the number of sides, and
- P be the number of plane angles.

By naming each component, Euler observed some general relationships that occur for all polyhedron. For example, the definition of a side leads to the equation 2S = E, where S is the number of sides and E is the number of edges. It can then be stated that the number of edges must be an even number, for if this were not the case, it would be possible to have a half of a side, which contradicts the definition of S.

By looking at the condition to form a face, we can derive another formula. Since a face is a surface enclosed by at least 3 sides, we can state that $S \geq 3F$. Furthermore, there must be at least 3 faces surrounding every vertex ($F \geq 3V$). Additionally, it can be stated that the number of plane angles equals the number of sides, P = S, that $2E \geq 3F$, and that $2E \geq 3V$

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

In summary: $2S = E$

$$S \geq 3F$$

$$F \geq 3V$$

$$P = S$$

$$2E \geq 3F$$

$$2E \geq 3V$$

The Definition of a Polyhedron

There are many definitions of a polyhedron. The definition used in this project given by P. Cromwell in his book entitled "Polyhedra" is the following:

"A *polyhedron* is the union of a finite set of polygons such that

i. Any pair of polygons meet only at their sides or corners.

ii. Each side of each polygon meets exactly one other polygon along an edge.

iii. It is possible to travel from the interior of any polygon to the interior of another.

iv. Let V be any vertex and lef F_1, F_2, …, F_n be the n polygons which meet at V. It is possible to travel over the polygons F_i from one to any other without passing through V."

Euler's Own Proof

1. Explanation

Although Euler presented the formula, he was unable to prove his result absolutely. His proof is based on the principle that polyhedrons can be truncated. Euler proceeds by starting with a polyhedron consisting of a large number of vertices, faces, and edges. By removing a vertex, you remove at least 3 faces (while exposing a new face), and at least 3 edges. As you continue, more vertices are removed, until eventually you will find that Euler's proof degenerates into an object that is not a polyhedron. A polyhedron must consist of at least 4 vertices. If there are less than 4 vertices present, a degenerate result will occur, and Euler's formula fails. While the proof fails to prove his formula, it does show that truncated and augmented platonic polyhedra satisfy the equation, (thereby including classes such as the Archimedean solids, and pyramids).

2. Proof in Pictures

Click here to see an example of Euler's proof using an octahedron.

LEGENDRE'S PROOF USING RADIAL PROJECTIONS

I. Explanation

Another interesting proof that has wider applications, is a proof credited to Adrien Marie Legendre. This proof inscribes a convex polyhedron inside a sphere, where the centre is found in the region enclosed by the polyhedron, and the polyhedron is fully inscribed. We then proceed to project light rays from the origin through each of the vertices on the polyhedron. By finding where these rays intersect the sphere, and connecting the points of intersection by the arcs that characterize the shortest distance between two points along the sphere, we produce a radial projection of the polyhedron. Vertices are now the points where the arcs meet, edges are now arcs, and faces are now spherical polygons. By combining the characteristics of a sphere, spherical polygons, and the summarized characteristics of vertices, edges, and faces (from above), Euler's formula can be derived.

Main facts:

- If you sum all of the angles surrounding a vertex, the resulting angle will be 2π. It follows that the total angle sum around all the vertices will be $2\pi V$.

- From above, $2S = E$.

- Each side has an angle of π.

- Each face has an angle of 2π.

- The surface area of a sphere of $R=1$ is 4π.

Let's now establish the equation for the surface area of a sphere. Let us assume the radius of the sphere is 1.

Area of a Spherical Polygon** = angle sum – sides on single face (π) + 2

Surface Area = (# of spherical polygons mapped on the sphere) (Area of Spherical Polygon)

Surface Area = V(angle sum) - $S\pi + 2\pi F$

$4\pi = 2\pi V - 2E\pi + 2\pi F$

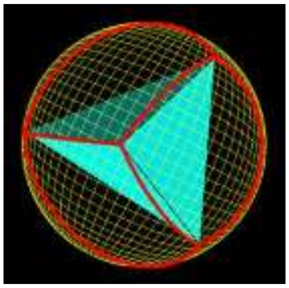
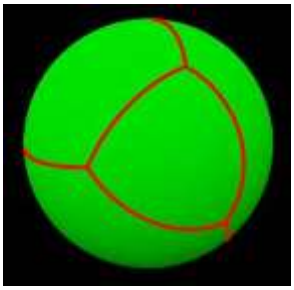
If we divide both sides of the equation by 2π, we get the formula:

$2 = V - E + F$

**This formula is from Girard's Theorem for spherical polygons, although no further discussion of this theorem is provided in this project.

II. Proof in Pictures

The following demonstrates elements of Legendre's proof of Euler's formula through pictures, by inscribing a tetrahedron inside a unit sphere. Click here to view it.

III. Difficulties

At first glance, it seemed like an easy task to solve for the path along the surface of a sphere that connected two points. However, upon attempting to code such a procedure, it turns out there are some interesting questions that need to be answered.

Let us first look at how to parameterize a unit sphere of form $x^2 + y^2 + z^2 = 1$. Let u, v be the parameters we introduce to describe this curve. This being said, any point on the sphere satisfies the equation

$(\cos(u)\cos(v), \cos(v)\sin(u), \sin(v))$ where $0^o \leq u, v \leq 360^o$.

However, knowing how to parameterize the unit sphere is unnecessary when trying to derive the direct path from any two points, A and B, because there is an additional condition that must be satisfied. To find the appropriate arc, we must find the plane that contains A, B, and the origin of the sphere. This plane will also contain all of the points that describe the path between the two points.

After looking at different cases, I found the easiest way to find the arc along the great circle was to calculate the midpoint between the two points we wish to connect. Let C be the centre of the sphere.

Let $A = (x_A, y_A, z_A)$, $B = (x_B, y_B, z_B)$, and $C = (0, 0, 0)$.

The direction of the midpoint is therefore M, where

$M = [\ ((x_A + x_B)/2), ((y_A + y_B)/2), ((z_A + z_B)/2)\ ]$.

To find the midpoint along the arc of the great circle, we must therefore find where the vector of the midpoint intersects $x^2 + y^2 + z^2 = 1$.

To do accomplish this, you can find the unit length M, and divide M by it. This will result in the midpoint. Therefore, the midpoint along a great arc, m, is:

$m = M / ||M||$

A PROOF BASED PRINCIPALLY ON ILLUSTRATIONS

i. Explanation

I found Von Staudt's proof to be one of the nicer proofs for Euler's formula. Von Staudt's proof begins by picking any vertex on a convex polyhedron. From this vertex, we look for a vertex that has never been coloured, and connect the vertices by colouring along the edge that connects them. Now at the new vertex, we look for another vertex that has never been coloured. If we find one, we connect the vertices and colour along the edge that connects them again. We proceed in this fashion until there are no more vertices to be visited.

If there are no more vertices to colour, then all vertices have been coloured. When traversing a polyhedron in this manner, we can see that this proof requires that there is always a path that touches all vertices only once.

Around the same time that this proof was being published, an Irish mathematician Sir William Rowan Hamilton (1805 – 1865) was examining the problem of finding a path in a graph, or along a solid, where each vertex is traversed only once. This path is called a Hamiltonian circuit, and finding whether or not a circuit exists in a figure is quite a challenge. To solve the problem on an arbitrary graph is known to be intractable; no efficient algorithm is known. Hamilton actually created this idea as a sort of puzzle, find the path to connect every vertex on a dodecahedron, visiting every vertex only once, and sold his dodecahedron puzzle to an Irish merchant who began to sell it. It turns out that it is quite a difficult task to identify whether or not a solid has a Hamiltonian circuit or not, and it may require a proof by exhaustion to determine whether or not a path is possible. However, it has been proven that all Platonic solids, Archimedean solids, and planar-4-connected graphs have Hamiltonian circuits.

As we begin our proof, we initially connect 2 vertices, and for every other edge we shade (let's use red), we find another vertex that we previously had not visited. Therefore, by counting the number of edges that we shade red we can determine the number of vertices. The formula is:

Red Edges = V – 1

Furthermore, Von Staudt states that all the vertices have been coloured. To prove this, assume this is not the case. Then there would be a vertex that is not coloured, which means we were not done colouring our edges red.

Next we will begin by examining faces. Pick a face and shade it green, as well as any edges that have not yet been coloured red. Proceed by finding another face who has only one green side, and shade green as before. Continue until there are either no more faces to colour that satisfy the condition. Since it is impossible that a face exists with all four sides coloured, all faces must be shaded green. The relationship between green edges and faces can be described as:

Green Edges = F – 1

The total number of edges, E, will equal the number of green edges and red edges.

E = Red Edges + Green Edges

E = V – 1 + F – 1

Therefore,

E = V + F – 2

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Proof in Pictures

I have illustrated Von Staudt's proof for a cube initially to clearly demonstrate how the proof is to proceed. Next, the same proof is illustrated on a dodecahedron.

CONCLUSION

For over two centuries people have discussed, proven and disproved Euler's formula. The sample proofs I have provided by no means represent the sole representations of Euler's formula. It is possible to find proofs based on electrical charge that will prove the formula. There are others that use Jordan curves and planar graphs, and there are other proofs that rely on topology. Furthermore, by carefully declaring the case and family of solids you wish to explain, the formula V + F – 2 = E can be generalized (by changing the 2) to explain more topologically complicated solids. (L'Huilier, a Swiss mathematician, investigated such figures and discovered that this was possible.) It is this variety of proofs and applications of Euler's formula that makes it such an interesting topic to study.

CONCEPT OF MEASUREMENT

MEASUREMENT

Measurement is a process of finding a number that shows the amount of something.

Measurement, the process of associating numbers with physical quantities and phenomena. Measurement is fundamental to the sciences; to engineering, construction, and other technical fields; and to almost all everyday activities. For that reason, the elements, conditions, limitations, and theoretical foundations of measurement have been much studied. See also measurement system for a comparison of different systems and the history of their development.

Measurements may be made by unaided human senses, in which case they are often called estimates, or, more commonly, by the use of instruments, which may range in complexity from simple rules for measuring lengths to highly sophisticated systems designed to detect and measure quantities entirely beyond the capabilities of the senses, such as radio waves from a distant star or the magnetic moment of a subatomic particle.

Measurement begins with a definition of the quantity that is to be measured, and it always involves a comparison with some known quantity of the same kind. If the object or quantity to be measured is not accessible for direct comparison, it is converted or "transduced" into an analogous measurement signal. Since measurement always involves some interaction between the object and the observer or observing instrument, there is always an exchange of energy, which, although in everyday applications is negligible, can become considerable in some types of measurement and thereby limit accuracy.

MEASUREMENT INSTRUMENTS AND SYSTEMS

In general, measuring systems comprise a number of functional elements. One element is required to discriminate the object and sense its dimensions or frequency. This information is then transmitted throughout the system by physical signals. If the object is itself active, such as water flow, it may power the signal; if passive, it must trigger the signal by interaction either with an energetic probe, such as a light source or X-ray tube, or with a carrier signal. Eventually the physical signal is compared with a reference signal of known quantity that has been subdivided or multiplied to suit the range of measurement required. The reference signal is derived from objects of known quantity by a process called calibration. The comparison may be an analog process in which signals in a continuous dimension are brought to equality.

An alternative comparison process is quantization by counting, i.e., dividing the signal into parts of equal and known size and adding up the number of parts.

Other functions of measurement systems facilitate the basic process described above. Amplification ensures that the physical signal is strong enough to complete the measurement. In order to reduce degradation of the measurement as it progresses through the system, the signal may be converted to coded or digital form. Magnification, enlarging the measurement signal without increasing its power, is often necessary to match the output of one element of the system with the input of another, such as matching the size of the readout meter with the discerning power of the human eye.

One important type of measurement is the analysis of resonance, or the frequency of variation within a physical system. This is determined by harmonic analysis, commonly exhibited in the sorting of signals by a radio receiver. Computation is another important measurement process, in which measurement signals are manipulated mathematically, typically by some form of analog or digital computer. Computers may also provide a control function in monitoring system performance.

Measuring systems may also include devices for transmitting signals over great distances (see telemetry). All measuring systems, even highly automated ones, include some method of displaying the signal to an observer. Visual display systems may comprise a calibrated chart and a pointer, an integrated display on a cathode-ray tube, or a digital readout. Measurement systems often include elements for recording. A common type utilizes a writing stylus that records measurements on a moving chart. Electrical recorders may include feedback reading devices for greater accuracy.

The actual performance of measuring instruments is affected by numerous external and internal factors. Among external factors are noise and interference, both of which tend to mask or distort the measurement signal. Internal factors include linearity, resolution, precision, and accuracy, all of which are characteristic of a given instrument or system, and dynamic response, drift, and hysteresis, which are effects produced in the process of measurement itself. The general question of error in measurement raises the topic of measurement theory.

MEASUREMENT THEORY

Measurement theory is the study of how numbers are assigned to objects and phenomena, and its concerns include the kinds of things that can be measured, how different measures relate to each other, and the problem of error in the measurement process. Any general theory of measurement must come to grips with three basic problems: error; representation, which is the justification of number assignment; and uniqueness, which is the degree to which the kind of representation chosen approaches being the only one possible for the object or phenomenon in question.

Various systems of axioms, or basic rules and assumptions, have been formulated as a basis for measurement theory. Some of the most important types of axioms include axioms of order, axioms of extension, axioms of difference, axioms of conjointness, and axioms of geometry. Axioms of order ensure that the order imposed on objects by the assignment of numbers is the same order attained in actual observation or measurement. Axioms of extension deal with the representation of such attributes as time duration, length, and mass, which can be combined, or concatenated, for multiple objects exhibiting the attribute in question. Axioms of difference govern the measuring of intervals. Axioms of conjointness postulate that attributes that cannot be measured empirically (for example, loudness, intelligence, or hunger) can be measured by observing the way their component dimensions change in relation to each other. Axioms of geometry govern the representation of dimensionally complex attributes by pairs of numbers, triples of numbers, or even *n*-tuples of numbers.

The problem of error is one of the central concerns of measurement theory. At one time it was believed that errors of measurement could eventually be eliminated through the refinement of scientific principles and equipment. This belief is no longer held by most scientists, and almost all physical measurements reported today are accompanied by some indication of the limitation of accuracy or the probable degree of error. Among the various types of error that must be taken into account are errors of observation (which include instrumental errors, personal errors, systematic errors, and random errors), errors of sampling, and direct and indirect errors (in which one erroneous measurement is used in computing other measurements).

Measurement theory dates back to the 4th century BC, when a theory of magnitudes developed by the Greek mathematicians Eudoxus of Cnidus and Thaeatetus was included in Euclid's *Elements*. The first systematic work on observational error was produced by the English mathematician Thomas Simpson in 1757, but the fundamental work on error theory was done by two 18th-century French astronomers, Joseph-Louis Lagrange and Pierre-Simon Laplace. The first attempt to incorporate measurement theory into the social sciences also occurred in the 18th century, when Jeremy Bentham, a British utilitarian moralist, attempted to create a theory for the measurement of value. Modern axiomatic theories of measurement derive from the work of two German scientists, Hermann von Helmholtz and Otto Hölder, and contemporary work on the application of measurement theory to psychology and economics derives in large part from the work of Oskar Morgenstern and John von Neumann.

Since most social theories are speculative in nature, attempts to establish standard measuring sequences or techniques for them have met with limited success. Some of the problems involved in social measurement include the lack of universally accepted theoretical frameworks and thus of quantifiable measures, sampling errors, problems associated with the intrusion of the measurer on the object being measured, and the subjective nature of the information received from human subjects. Economics is probably the social science that has had the most success in adopting measurement theories, primarily because many economic variables (like price and quantity) can be measured easily and objectively. Demography has successfully employed measurement techniques as well, particularly in the area of mortality tables.

EXAMPLES OF MEASUREMENT

Time

The ongoing sequence of events is time. We can measure time in seconds, minutes, hours, days, weeks, months, and years.

A clock and a calendar help us to measure time.

Weight

The amount of matter a thing consists of is called its weight. Measuring weight means to measure the heaviness of a thing.

Weight can be measured in grams, kilograms, and pounds.

Length

The amount of something that is measured from one end to the other along the longest side is called its length.

Length is measured in centimeters, meters, kilometers, feet, and miles

Capacity

Capacity is a measure of how much quantity a thing can hold.

Capacity is measured in liters and gallons

Temperature

The temperature of a thing is the measurement of how hot or cold it is.

Temperature is measured in Celsius, Fahrenheit, and Kelvin.

We can convert from one unit of measurement to another.

There are two measurement systems:438834709

The Metric System:

This system is based on the meter, liter, and gram as units of length (distance), capacity (volume), and weight (mass) respectively.

The US Standard Units:

This system uses inches, feet, yards, and miles to measure length or distance.

Capacity or volume is measured in fluid ounces, cups, pints, quarts or gallons.

Weight or mass is measured in ounces, pounds and tons.

Fun Facts:

- The metric system was first proposed by the French astronomer and mathematician Gabriel Mouton in 1670 and was standardized in Republican France in the 1790s.
- The US Standard units are also known as "English Units" or "US Customary Units".

USING NON-STANDARD AND STANDARD UNITS OF MEASUREMENT;

Standard units are common units of measurement such as centimetres, grams and litres. Non-standard measurement units might include cups, cubes or sweets.

Standard units are what we usually used to measure things like weight, length and volume. Standard units that would be introduced in primary school are grams, kilograms, meters, kilometres, millilitres and litres.

Non-standard units are used by children in EYFS and Year 1 to introduce without having to use scales of any kind as this can make it seem more complicated.

Although standard measurements are an essential part of the 2014 National Curriculum for Maths, non-standard measurements are often used with EYFS and Year one students to help them grasp the concept of measurements.

HOW ARE NON-STANDARD UNITS USED IN THE CLASSROOM?

Children are first introduced to the concept of measurement during the EYFS stage of their learning. However, children at this stage don't read any scales - instead, they usually begin to measure everyday items on how they feel (light or heavy). You could ask children to compare items by asking them: do you think the pencil or sharpener is heavier? This is a great way to get them thinking about measurement and how we determine the difference between items.

Once children reaching KS1 students are prompted to measure using centimetres or kilograms, it is common for teachers to introduce them to the concept and skills of measuring using non-standard units.

These are often practical, easy to visualise examples which can be demonstrated in the classroom. For instance, how many cups of sand are needed to fill a bucket or how many biscuits will fit into a tin.

This will allow students to develop their counting skills as well as get used to the language around measurements, such as “x units got into y unit”. This creates a strong foundation of knowledge which will help them as they begin to learn standard units of measurement.

USING STANDARD UNITS OF MEASUREMENT

Children begin to use standard units in Year 2. They'll begin to learn and understand the different equipment needed to measure different items. For example, they'd need to know whether to measure items using centimetres, kilograms or metres.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Once children reaching Year 3 they'll carry on working on everything they've learnt in Year 2, but at more advanced levels. Year 2 children will begin to problem-solve using practical ways such as scales. They will also begin to understand the different conversion measurements and should be able to recall the following:

- 1 litre = 1000 millilitres.
- 1 metre = 100 centimetres.
- 1 kilogram = 1000 grams.

In year 4 children will start to convert and should be able to determine different measurements. For example, children should know that 1.4 litres is the same as 1400 ml. This will help them with problem-solving when sometimes children must find out a measurement and convert it to a standard unit.

In Year 5, children will solve problems using measurement further but will also use addition, subtraction, multiplication and division to achieve the answer.

Once children reach the last year of Primary education (Year 6) they will continue problem solving using the four operations, whilst converting between units using decimals.

EXAMPLES OF STANDARD AND NON-STANDARD UNITS

The following are all standard units of measurement. Though they may not be common, they are still accepted units of measurement across the world.

- Centimetres (1 cm = 0.39 Inches)
- Feet (1 Foot = Approx. 30 cm)
- Kilograms (1 Kilogram = 2.2. Pounds)
- Cups (1 Cup = 10 Fluid Ounces)
- Hands (1 Hand = 4 inches)

These units of measurement, however, are non-standard and not generally accepted or known by other people. They are likely only used in a classroom situation or in a casual context. They may give only a very approximate measurement - more like an estimate than a fixed measurement.

- Heads (e.g. Megan is a head taller than her sister)
- Sweets (e.g. 20 sweets in a bag)
- Squares (e.g. 15 squares of chocolate in a bar)
- Phone Book (e.g. This is heavier than a stack of Phone Books!)
- Stone’s Throw (e.g. It’s not far - it’s only a stone's throw away.)

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

HOW CAN I TEACH STANDARD AND NON-STANDARD UNITS TO MY CLASS?

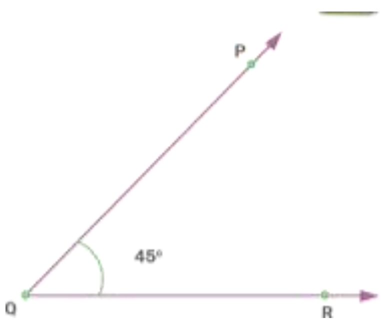
There are a number of National Curriculum aims relating to measurements when it comes to KS1 Maths. It is important that students have a grounding in both standard and non-standard units of measurement.

TYPES OF ANGLES

When two lines intersect, at the point of their intersection an angle is formed. Learning about angles is important, as they form the base of geometry. The two rays that form the angle are known as the sides of the angle. Also, it is not necessary that an angle is formed by intersection of two straight lines; it can be formed by intersection of two curved lines too.

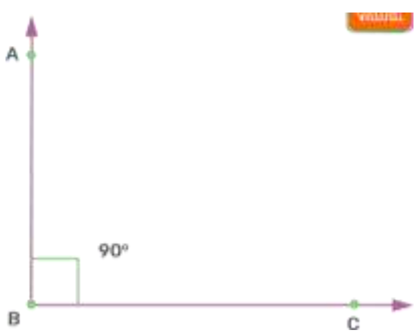
1. Acute Angle

An angle which measures less than 90° is called an acute angle. The measure between 0° to 90°. In the picture below, the angle formed by the intersection of PQ and QR at Q forms an angle PQR which measures 45°. Thus, PQR is called an acute angle.



2. Right Angle

An angle which measures exactly 90° is called a right angle. It is generally formed when two lines are perpendicular to each other. In the figure below, line AB intersects line BC at B and form an angle ABC which measures 90°.

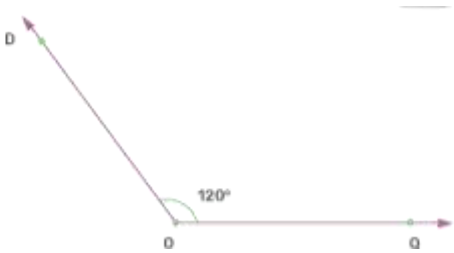


3. Obtuse Angle

An angle that measures greater than 90° is known as the obtuse angle. The angle measure ranges from 90° to 180°. An obtuse angle can also be found out if we have the measure of the acute angle.

Obtuse Angle Measure = (180 - acute angle measure)

In the picture above, line segment DO intersects line segment OQ at point O and forms an angle DOQ measuring 120°. Thus, it is an obtuse angle.



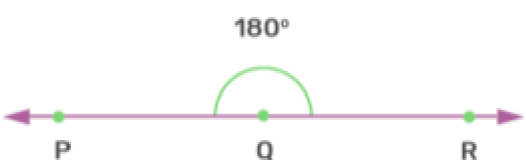
Also, if we extend line OQ to OP then we can find a measure of the acute angle.

DOP = 180° - DOQ = 180° - 120° = 60°

1. Straight Angle

The angle which measures exactly 180° is called a straight angle. This is similar to a straight line, thus the name straight angle.

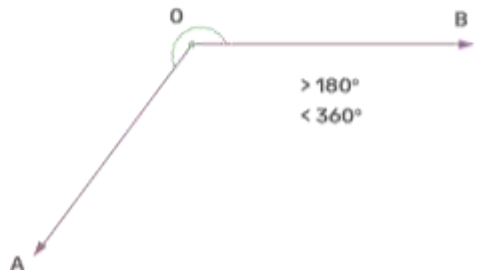
A straight angle is nothing but a mixture of an obtuse angle and acute angle on a line.



2. Reflex Angle

The angle which measures greater than 180° and less than 360° is known as the reflex angle. The reflex angle can be calculated if the measure of the acute angle is given, as it is complementary to the acute angle on the other side of the line.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)



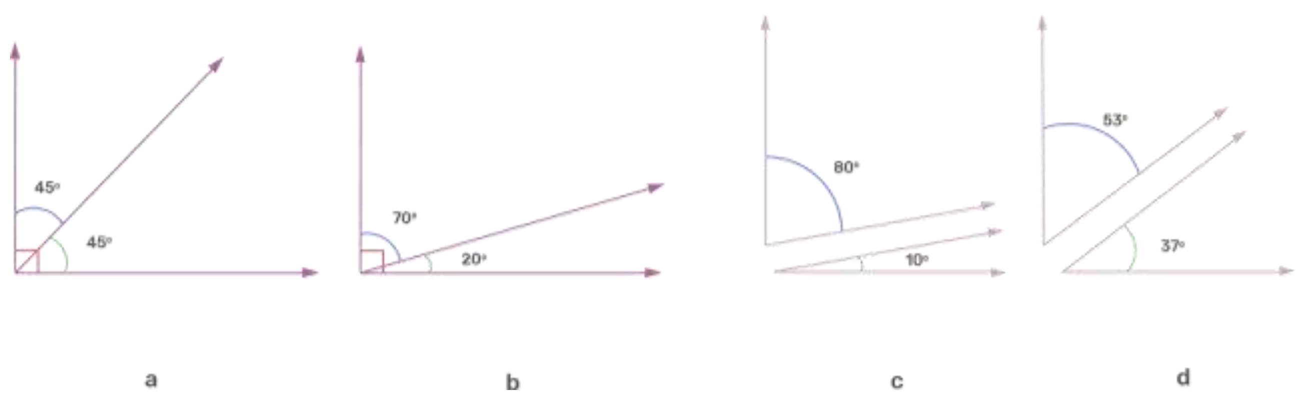
Using the reflex angle, we can find the measure of the acute angle.

A Measure of Acute Angle = 360° – a Measure of Reflex Angle

Complementary & SupplementaryAngle

1. Complementary Angle

If two angles add up to measure 90° then they are known as complementary angles. The angles don't have to be adjacent to each other to be known as complementary. As long as they add up to 90° they will be known as complementary angles.



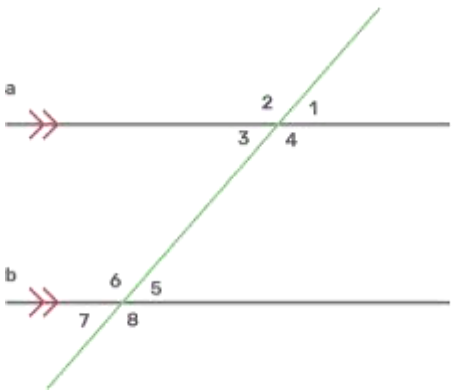
In the figure, a and b the angles are present adjacent to each other and add up to 90° and thus are known as complementary angles. In figure c and d, the angles are not adjacent to each other, but they add up to 90° and thus they are known as complementary angles.

2. Supplementary Angles

When two angles add up to 180° then they are known as supplementary angles. There are various types of supplementary angles.

a. Vertical Angles

Angles which have a common vertex and the sides of the angle are formed by the same lines are known as vertical angles. Vertical angles are equal to each other.



In the above figure, 1 and 3, 2 and 4, 6 and 8 and 5 and 7 are vertical angles. Also, 3, 4,5, 6 are known as interior angles and 1,2,7,8 are known as exterior angles.

b. Alternate Interior Angles

These are a pair of interior angles present on the opposite side of the transversal. The easiest way to spot alternate interior angles is to identify a "Z" on the interior side.

In the above figure, 3 and 5, 4 and 6 are interior angles. The interior angles equal to one another.

c. Alternate Exterior Angles

This is similar to alternate interior angles; just that it is present on the exterior side. In the above figure, 1 and 7, 2 and 8 are the pair of alternate exterior angles. Similar to alternate interior angles, even alternate exterior angles are equal to one another.

d. Corresponding Angles

Angles which are present in a similar position are known as corresponding angles.

Adjacent angles have the same vertex and arm.

e. Adjacent angles have the same vertex and arm.

f. Zero Angle

A zero angle (0°) is an angle formed when both the angle's arms are at the same position.

∠ RPQ = 0° (zero angle)

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

g. Complete Angle

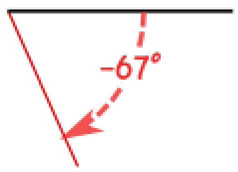
A complete angle is equal to 360°. 1 revolution is equal to 360°.

Positive and Negative Angles

When measuring from a line:

- a **positive** angle goes counterclockwise (opposite direction that clocks go)
- a **negative** angle goes clockwise

Example: −67°



Summary

Angle Type	Angle measure
Acute angle	Greater than 0 °, Less than 90°
Right angle	90°
Obtuse angle	Greater than 90°, less than 180°
Straight angle	180°
Reflex angle	Greater than 180°, less than 360°

The major basis of geometry is angles. Angles finds its application in nearly all types of questions, be it trigonometry to closed shapes. Understanding angles and angle types will help in solving a lot of tricky questions. Thus, make sure that you understand it well.

PARTS OF AN ANGLE

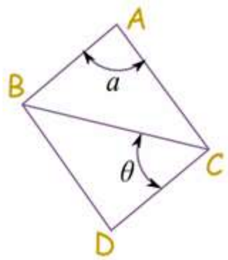
The corner point of an angle is called the **vertex**

And the two straight sides are called **arms**

The angle is the *amount of turn* between each arm.

HOW TO LABEL ANGLES

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)



There are two main ways to label angles:

1. give the angle a name, usually a lower-case letter like **a** or **b**, or sometimes a Greek letter like **α** (alpha) or **θ** (theta)

2. or by the three letters on the shape that define the angle, with the middle letter being where the angle actually is (its vertex).

Example angle "**a**" is "**BAC**", and angle "**θ**" is "**BCD**"

FINDING PERIMETER AND AREAS OF TRIANGULAR

PERIMETER OF A TRIANGLE

The perimeter of any two-dimensional figure refers to a distance around the figure. We can compute the perimeter of any closed figure by just summing up the length of all the three sides. In this section, you will learn how to calculate the perimeter of different types of triangles when there lengths are known.

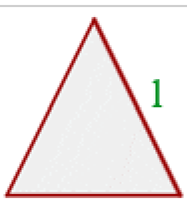
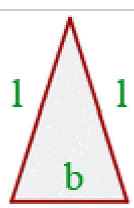
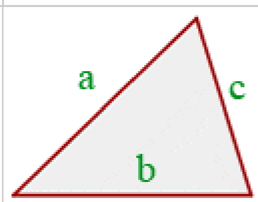
Formula

The formula for finding the perimeter of a triangle is given below:

Perimeter = Sum of three sides

The standard unit of perimeter is meter or centimeter.

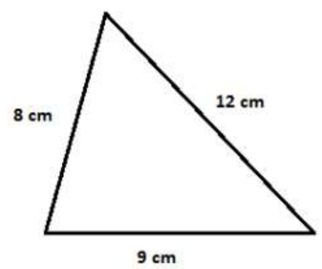
The following table shows the formula for computing the sides of three types of triangle:

Equilateral Triangle	**Isosceles Triangle**	**Scalene Triangle**
$P = 3 \cdot l$	$P = 2 \cdot l + b$	$P = a + b + c$
		

- It shows that the perimeter of an equilateral triangle is 3 times its length
- The perimeter of an isosceles triangle is the sum of 2 times length of the equal sides and base

- The perimeter of the scalene triangle is equal to the sum of the lengths of all three sides of the triangle

Consider the following triangle whose lengths of three sides are given:



The length of the sides are 12cm 9 cm and 8 cm respectively. To calculate the perimeter, substitute these values in the formula below:

Perimeter = a + b + c

= 12 + 9 + 8 = 29 cm

AREA OF A TRIANGLE

An area of a triangle is defined as follows:

The total region that is enclosed by three sides of a triangle is known as an area of that triangle

This area of a triangle is equal to the half of the product of base and height of a triangle. Mathematically, we can write the formula of area of a triangle like this:

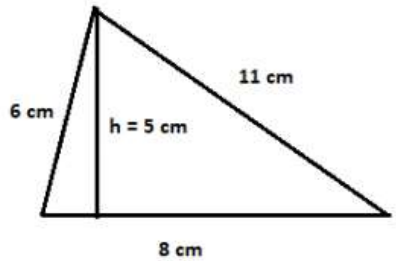
Area = $\frac{1}{2} \times base \times height$

It means that to calculate the area of a triangle, we should know the base and height of the triangle. It should be kept in mind that the base and height of a triangle are perpendicular to each other. Sometimes, the height of a triangle is unknown. To find the area of such a triangle, we need to know the height first. In these situations, we different formulas to compute the areas of triangle. All these scenarios are discussed below with examples.

AREA OF A TRIANGLE WHEN HEIGHT IS KNOWN

In this section, we will see how to calculate the area of a triangle with known height.

Find the area of the following triangle:



In the above triangle, height and base of the triangle is given, so we will use the general formula for finding the area of the triangle.

Area = $\frac{1}{2} \times base \times height$

Area = $\frac{1}{2} \times 8 \times 6$

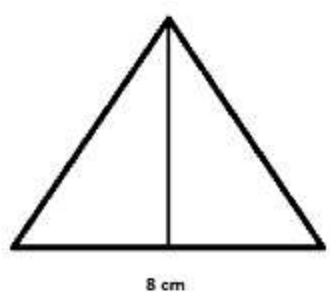
Area = $24cm^2$

AREA OF AN EQUILATERAL TRIANGLE (HEIGHT UNKNOWN)

All sides of an equilateral triangle are equal. The formula to calculate the area of an equilateral triangle is given below:

Area = $\frac{\sqrt{3}}{4}l^2$

For example, consider an equilateral triangle below whose measurement of one side is given:



We will substitute 8 in the above formula to calculate the area of an equilateral triangle:

Area = $\frac{\sqrt{3}}{4}(8)^2$

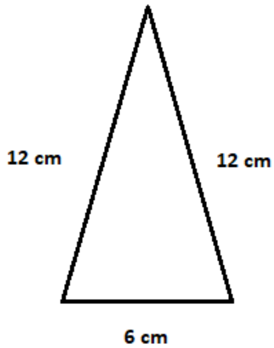
Area = $\frac{\sqrt{3}}{4}64$

Area = $27.71cm^2$

AREA OF AN ISOSCELES TRIANGLE WITH KNOWN SIDES ONLY

Isosceles triangle is a triangle whose length of two sides are equal. We can have a situation in which the sides of an isosceles triangle are known, but the height is unknown. In this situation, we can use the following formula to compute the area of a triangle:

Area = $\frac{1}{2}[\sqrt{a^2 - \frac{b^2}{4}}) \times b]$



In the above triangle, the lengths of two sides are equal, hence it is an isosceles triangle. We do not know the height of the triangle, so we will use the above discussed formula to compute its area.

Area = $\frac{1}{2}[\sqrt{a^2 - \frac{b^2}{4}}) \times b]$

Area = $\frac{1}{2}[\sqrt{12^2 - \frac{6^2}{4}}) \times 6]$

Area = $34.8569m^2$

We know that remembering all the above formulas for equilateral or isosceles can be difficult. When sides are known and height is unknown, you can simply use the Heron's formula which is discussed below to calculate the area of a triangle. This is true for all types of triangles whether they are equilateral, isosceles, right angle or squalene.

HERON'S FORMULA TO CALCULATE THE AREA WHEN SIDES ARE KNOWN ONLY

We can also use the Heron's formula to calculate the area of a triangle when the length of three sides are known. To employ this formula, we should know the perimeter of a triangle which refers to the distance covered around the triangle and is computed by summing up the length of all three sides of the triangle. Use the following steps to calculate the area using Heron's formula:

- In the first step of the process, find the semi perimeter of the triangle by summing up the three sides and dividing the figure by 2. Semi perimeter is simply half of the perimeter of a triangle.
- In the second step, apply the value obtained in the first step of the triangle in the main formula which is known as Heron's formula.

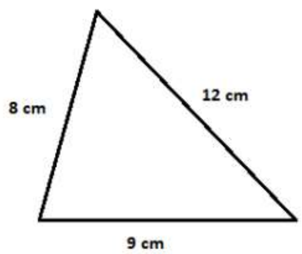
The Heron's formula to calculate the area of a triangle is given below:

Area = $\sqrt{s(s-a)(s-b)(s-c)}$

Here, s is the semi perimeter and a, b and c represent the sides of the triangle.

Consider the following example:

Find the area of the following triangle:



The values of sides a, b and c are 12 cm, 9 cm and 8 cm. First we will calculate the semi perimeter by substituting these values in the below formula:

Semi perimeter = $\frac{a+b+c}{2}$

Semi perimeter = $\frac{12+8+9}{2}$

Semi perimeter = $\frac{29}{2}$

$= 14.5cm$

Now, we will substitute this value of semi perimeter in the below formula to find the area:

Area = $\sqrt{s(s-a)(s-b)(s-c)}$

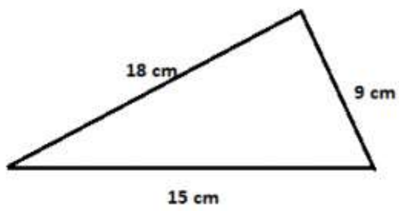
Area = $\sqrt{14.5(14.5-12)(14.5-9)(14.5-8)}$

Area = $35.991cm^2$

Consider another example below

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Find the area of the following triangle:



The values of sides a, b and c are 18 cm, 15 cm and 9 cm. First we will calculate the semi perimeter by substituting these values in the below formula:

Semi perimeter = $\frac{a+b+c}{2}$

Semi perimeter = $\frac{18+15+9}{2}$

Semi perimeter = $\frac{42}{2}$

$= 21cm$

Now, we will substitute this value of semi perimeter in the below formula to find the area:

Area = $\sqrt{s(s-a)(s-b)(s-c)}$

Area = $\sqrt{21(21-18)(21-15)(21-9)}$

Area = $\sqrt{21(3)(6)(12)}$

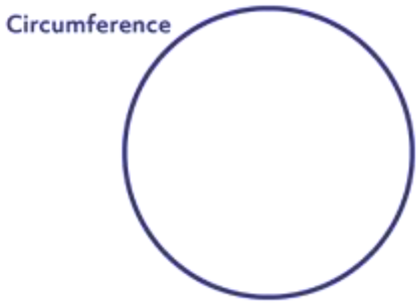
Area = $67.3498cm^2$

CIRCUMFERENCE AND AREA OF CIRCULAR REGIONS

WHAT IS THE CIRCUMFERENCE OF A CIRCLE?

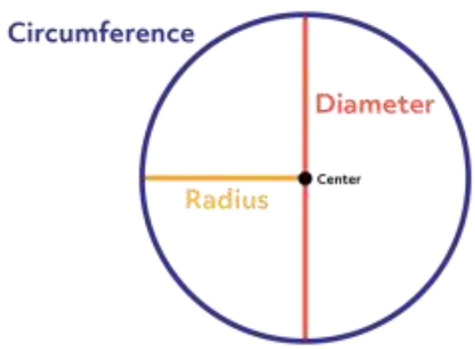
The **circumference** of a circle is the distance around the outside of the circle. It is like the **perimeter** of other shapes like squares. You can think of it as the line that defines the shape. For shapes made of straight edges this line is called the **perimeter** but for circles this defining line is called the **circumference.**

This diagram shows the circumference of a circle.



There are two other important distances on a circle, the **radius (r)** and the **diameter (d).** The radius, the diameter, and the circumference are the three defining aspects of every circle. Given the radius or diameter and pi you can calculate the circumference. The diameter is the distance from one side of the circle to the other at its widest points. The diameter will always pass through the center of the circle. The radius is half of this distance. You can also think of the radius as the distance between the center of the circle and its edge.

This diagram shows the circumference, diameter, center, and radius on a circle.



HOW CAN YOU CALCULATE THE CIRCUMFERENCE OF A CIRCLE?

If you know the diameter or radius of a circle, you can work out the circumference. To begin with, remember that pi is an irrational number written with the symbol π. π is roughly equal to 3.14.

The formula for working out the circumference of a circle is:

CIRCUMFERENCE OF CIRCLE = Π X DIAMETER OF CIRCLE

This is typically written as $C = \pi d$. This tells us that the circumference of the circle is three "and a bit" times as long as the diameter. We can see this on the graphic below:

You can also work out the circumference of a circle if you know its radius. Remember that the diameter is double the length of the radius. We already know that $C = \pi d$. If r is the radius of the circle, then $d = 2r$. So, $C = 2\pi r$.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Example 1

If a circle has a diameter of 10cm, what is its circumference?

Answer

We know that C = πd. Since the diameter is 10cm, we know that C = π x 10cm = 31.42cm (to 2 decimal places).

Example 2

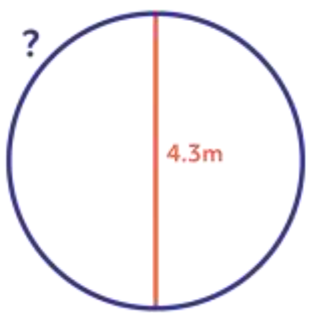
If a circle has a radius of 3m, what is its circumference?

Answer

We know that C = 2πr. Since the radius is 3m, we know that C = π x 6m = C= 18.84m (to 2 decimal places).

Example 3

Find the missing length (marked with a ?) on the diagram below:



Answer

The missing length is the circumference. Using the knowledge that the diameter is 4.3m on the diagram and knowing that C = πd, we can calculate the circumference. With a little thinking we can easily figure out that, C = π x 4.3m = 13.51m (to 2 decimal places). The missing length is 13.51m.

HOW TO CALCULATE THE CIRCUMFERENCE OF THE EARTH?

Have you ever wondered how big the earth is? Well, using pi it's possible to work out the circumference of the Earth! Scientists have discovered that the diameter of the Earth is 12,742km. Given this information, what is the circumference of the Earth? Get out a piece of paper and a calculator and see if you can work it out on your own.

Again, we know that C = πd, and that the diameter of the earth is 12,742km. Using this information, we can calculate the circumference of the Earth as C = π x 12,742km = 40,030km.

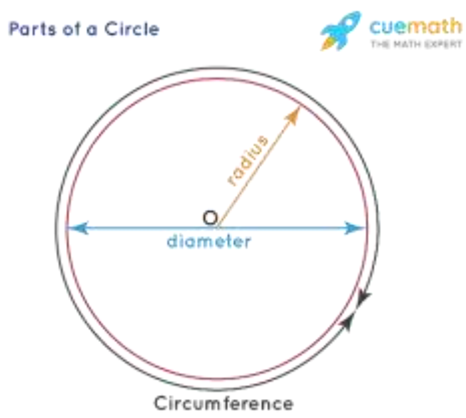
AREA OF A CIRCLE

The area of a circle is the space occupied by the circle in a two-dimensional plane. Alternatively, the space occupied within the boundary/circumference of a circle is called the area of the circle. The formula for the area of a circle is $A = \pi r^2$, where r is the radius of the circle. The unit of area is the square unit, for example, m^2, cm^2, in^2, etc. **Area of Circle = πr^2 or $\pi d^2/4$ in square units, where (Pi) π = 22/7 or 3.14.** Pi (π) is the ratio of circumference to diameter of any circle. It is a special mathematical constant.

The area of a circle formula is useful for measuring the region occupied by a circular field or a plot. Suppose, if you have a circular table, then the area formula will help us to know how much cloth is needed to cover it completely. The area formula will also help us to know the boundary length i.e., the circumference of the circle. Does a circle have volume? No, a circle doesn't have a volume. A circle is a two-dimensional shape, it does not have volume. A circle only has an area and perimeter/circumference. Let us learn in detail about the area of a circle, surface area, and its circumference with examples.

Circle and Parts of a Circle

A circle is a collection of points that are at a fixed distance from the center of the circle. A circle is a closed geometric shape. We see circles in everyday life such as a wheel, pizzas, a circular ground, etc. The measure of the space or region enclosed inside the circle is known as the area of the circle.



Radius: The distance from the center to a point on the boundary is called the radius of a circle. It is represented by the letter 'r' or 'R'. Radius plays an important role in the formula for the area and circumference of a circle, which we will learn later.

Diameter: A line that passes through the center and its endpoints lie on the circle is called the diameter of a circle. It is represented by the letter 'd' or 'D'.

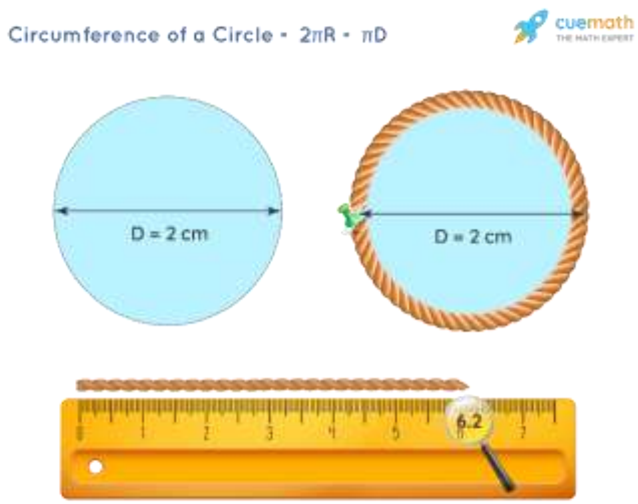
Diameter formula: The diameter formula of a circle is twice its radius. Diameter = 2 × Radius

d = 2r or D = 2R

If the diameter of a circle is known, its radius can be calculated as:

r = d/2 or R = D/2

Circumference: The circumference of the circle is equal to the length of its boundary. This means that the perimeter of a circle is equal to its circumference. The length of the rope that wraps around the circle's boundary perfectly will be equal to its circumference. The below-given figure helps you visualize the same. The circumference can be measured by using the given formula:



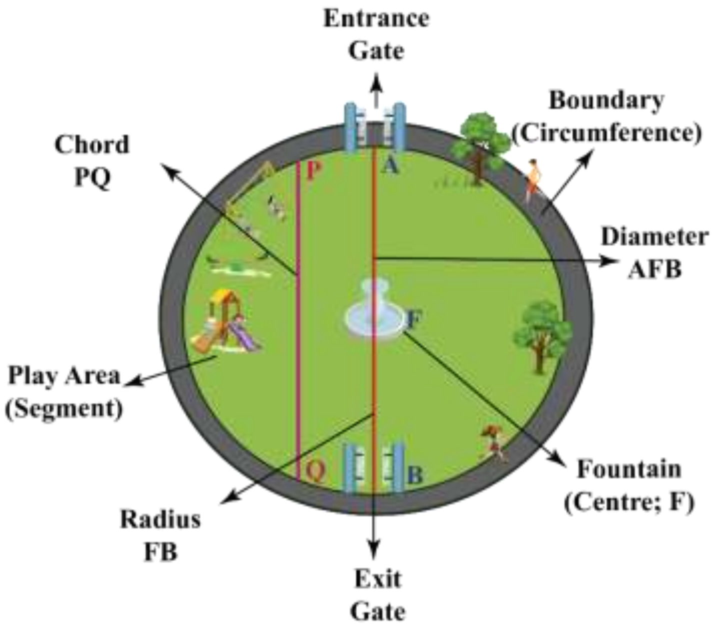
where 'r' is the radius of the circle and π is the mathematical constant whose value is approximated to 3.14 or 22/7. The circumference of a circle can be used to find the area of that circle.

For a circle with radius ‘r’ and circumference ‘C’:

- π = Circumference/Diameter
- $\pi = C/2r = C/d$
- $C = 2\pi r$

Let us understand the different parts of a circle using the following real-life example.

Consider a circular-shaped park as shown in the figure below. We can identify the various parts of a circle with the help of the figure and table given below.



In a Circle	**In our park**	**Named by the letter**
Centre	Fountain	F
Circumference	Boundary	
Chord	Play area entrance	PQ
Radius	Distance from the fountain to the Entrance gate	FA
Diameter	Straight Line Distance between Entrance Gate and Exit Gate through the fountain	AFB
Minor segment	The smaller area of the park, which is shown as the Play area	

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Major segment	The bigger area of the park, other than the Play area	
Interior part of the circle	The green area of the whole park	
Exterior part of the circle	The area outside the boundary of the park	
Arc	Any curved part on the circumference.	

WHAT IS THE AREA OF CIRCLE?

The area of a circle is the amount of space enclosed within the boundary of a circle. The region within the boundary of the circle is the area occupied by the circle. It may also be referred to as the total number of square units inside that circle.

AREA OF CIRCLE FORMULAS

The area of a circle can be calculated in intermediate steps from the diameter, and the circumference of a circle. From the diameter and the circumference, we can find the radius and then find the area of a circle. But these formulae provide the shortest method to find the area of a circle. Suppose a circle has a radius 'r' then the area of circle = πr^2 or $\pi d^2/4$ in square units, where π = 22/7 or 3.14, and d is the diameter.

Area of a circle, $A = \pi r^2$ square units

Circumference / Perimeter = $2\pi r$ units

Area of a circle can be calculated by using the formulas:

- Area = $\pi \times r^2$, where 'r' is the radius.
- Area = $(\pi/4) \times d^2$, where 'd' is the diameter.
- Area = $C^2/4\pi$, where 'C' is the circumference.

EXAMPLES USING AREA OF CIRCLE FORMULA

Let us consider the following illustrations based on the area of circle formula.

Example1: If the length of the radius of a circle is 4 units. Calculate its area.

Solution:
Radius(r) = 4 units(given)

Using the formula for the circle's area,
Area of a Circle = πr^2
Put the values,
$A = \pi 4^2$
$A = \pi \times 16$
$A = 16\pi \approx 50.27$

Answer: The area of the circle is 50.27 squared units.

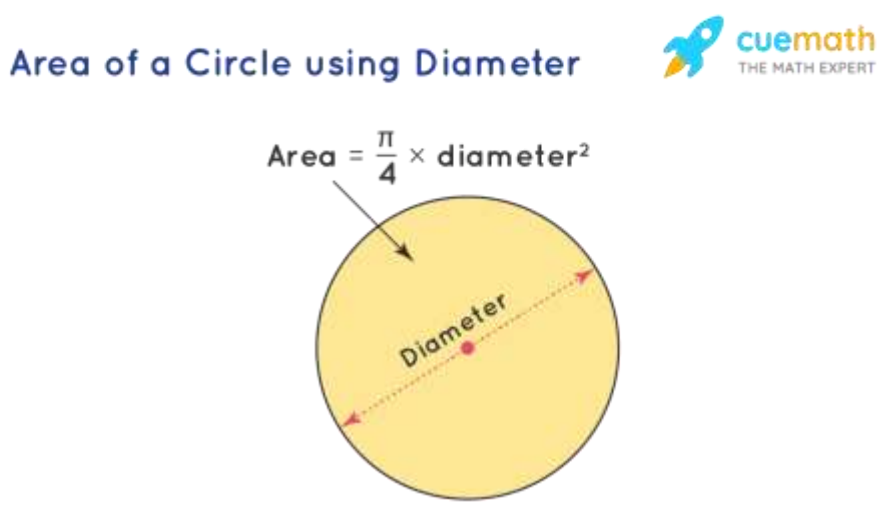
Example 2: The length of the largest chord of a circle is 12 units. Find the area of the circle.

Solution:
Diameter(d) = 12 units(given)
Using the formula for the circle's area,
Area of a Circle = $(\pi/4)\times d^2$
Put the values,
$A = (\pi/4) \times 12^2$
$A = (\pi/4) \times 144$
$A = 36\pi \approx 113.1$

Answer: The area of the circle is 113.1 square units.

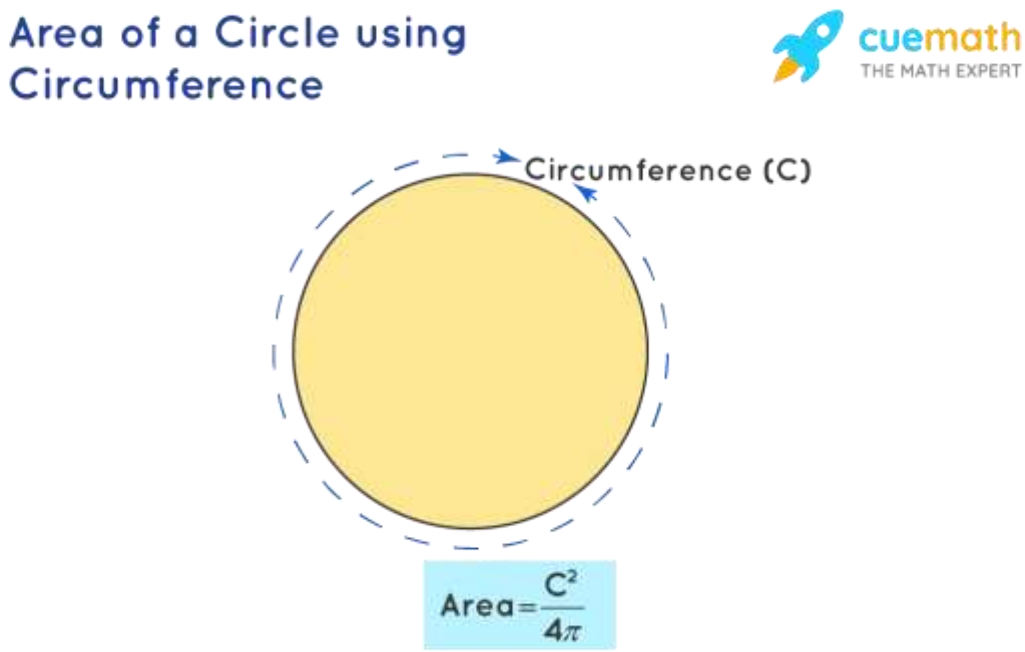
AREA OF A CIRCLE USING DIAMETER

The area of the circle formula in terms of the diameter is: Area of a Circle = $\pi d^2/4$. Here 'd' is the diameter of the circle. The diameter of the circle is twice the radius of the circle. d = 2r. Generally from the diameter, we need to first find the radius of the circle and then find the area of the circle. With this formula, we can directly find the area of the circle, from the measure of the diameter of the circle.



AREA OF A CIRCLE USING CIRCUMFERENCE

The area of a circle formula in terms of the circumference is given by the formula (Circumference)24π(Circumference)24π. There are two simple steps to find the area of a circle from the given circumference of a circle. The circumference of a circle is first used to find the radius of the circle. This radius is further helpful to find the area of a circle. But in this formulae, we will be able to directly find the area of a circle from the circumference of the circle.



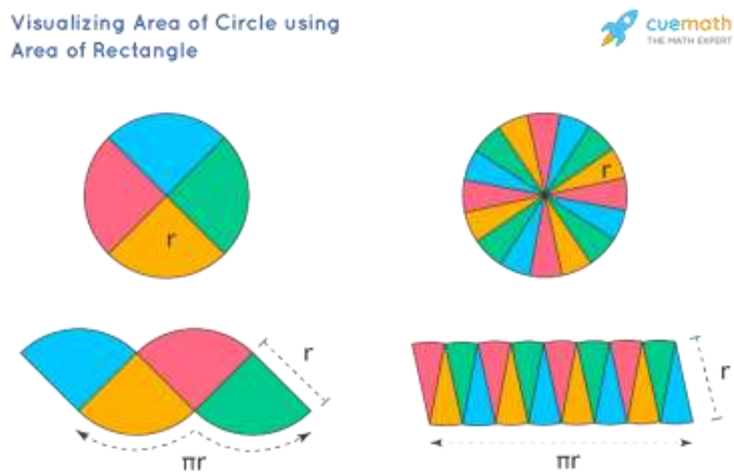
AREA OF A CIRCLE-CALCULATION

The area of the circle can be conveniently calculated either from the radius, diameter, or circumference of the circle. The constant used in the calculation of the area of a circle is pi, and it has a fractional numeric value of 22/7 or a decimal value of 3.14. Any of the values of pi can be used based on the requirement and the need of the equations. The below table shows the list of formulae if we know the radius, the diameter, or the circumference of a circle.

Area of a circle when the radius is known.	πr^2
Area of a circle when the diameter is known.	$\pi d^2/4$
Area of a circle when the circumference is known.	$C^2/4\pi$

DERIVATION OF AREA OF A CIRCLE

Why is the area of the circle is πr^2? To understand this, let's first understand how the formula for the area of a circle is derived.



Observe the above figure carefully, if we split up the circle into smaller sections and arrange them systematically it forms a shape of a parallelogram. When the circle is divided into even smaller sectors, it gradually becomes the shape of a rectangle. The more the number of sections it has more it tends to have a shape of a rectangle as shown above.

The area of a rectangle is = length × breadth

The breadth of a rectangle = radius of a circle (r)

When we compare the length of a rectangle and the circumference of a circle we can see that the length is = ½ the circumference of a circle

Area of circle = Area of rectangle formed = $\frac{1}{2}(2\pi r) \times r$

Therefore, the area of the circle is πr^2, where r, is the radius of the circle and the value of π is 22/7 or 3.14.

SURFACE AREA OF CIRCLE FORMULA

The surface area of a circle is the same as the area of a circle. In fact, when we say the area of a circle, we mean nothing but its total surface area. Surface area is the area occupied by the surface of a 3-D shape. The surface of a sphere will be spherical in shape but a circle is a simple plane 2-dimensional shape.

If the length of the radius or diameter or even the circumference of the circle is given, then we can find out the surface area. It is represented in square units. The surface area of circle formula $= \pi r^2$ where 'r' is the radius of the circle and the value of π is approximately 3.14 or 22/7.

REAL-WORLD EXAMPLE ON AREA OF CIRCLE

Ron and his friends ordered a pizza on Friday night. Each slice was 15 cm in length.

Calculate the area of the pizza that was ordered by Ron. You can assume that the length of the pizza slice is equal to the pizza's radius.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Solution:

A pizza is circular in shape. So we can use the area of a circle formula to calculate the area of the pizza.

Radius is 15 cm

Area of Circle formula = πr^2 = 3.14 × 15 × 15 = 706.5

Area of the Pizza = 706.5 sq. cm.

WORKED EXAMPLES

1. *Find the circumference and area of radius 7 cm.*

Solution:

Circumference of circle = $2\pi r$

= 2 × 22/7 × 7

= 44 cm

Area of circle = πr^2

= 22/7 × 7 × 7 cm^2

= 154 cm^2

2. *A race track is in the form of a ring whose inner circumference is 220 m and outer circumference is 308 m. Find the width of the track.*

Solution:

Let r_1 and r_2 be the outer and inner radii of ring.

Then $2\pi r_1 = 308$

$2 \times 22/7\ r_1 = 308$

$\Rightarrow r_1 = (308 \times 7)/(2 \times 22)$

$\Rightarrow r_1 = 49$ m

$2\pi r_2 = 220$

$\Rightarrow 2 \times 22/7 \times r_2 = 220$

$\Rightarrow r_2 = (220 \times 7)/(2 \times 22)$

$\Rightarrow r_2 = 35$ m

Therefore, width of the track = (49 - 35) m = 14 m

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

3. *The area of a circle is 616 cm². Find its circumference.*

Solution:

We know area of circle = πr^2

⇒ $22/7 \times r^2 = 616$

⇒ $r^2 = (616 \times 7)/22$

⇒ $r^2 = 28 \times 7$

⇒ $r = \sqrt{(28 \times 7)}$

⇒ $r = \sqrt{(2 \times 2 \times 7 \times 7)}$

⇒ $r = 2 \times 7$

⇒ r = 14 cm

Therefore, circumference of circle = $2\pi r$

= $2 \times 22/7 \times 14$

= 88 cm

4. *Find the area of the circle if its circumference is 132 cm.*

Solution:

We know that the circumference of circle = $2\pi r$

Area of circle = πr^2

Circumference = $2\pi r = 132$

⇒ $2 \times 22/7 \times r = 132$

⇒ $r = (7 \times 132)/(2 \times 22)$

⇒ r = 21 cm

Therefore, area of circle = πr^2

= $22/7 \times 21 \times 21$

= 1386 cm²

5. *The ratio of areas of two wheels is 25 : 49. Find the ratio of their radii.*

Solution:

If A_1 and A_2 are the area of wheels,

A_1/A_2 =25/49

⇒ $(\pi r_1^2)/(\pi r_2^2) = 25/49$

⇒ $(r_1^2)/(r_2^2) = 25/49$

⇒ $r_1/r_2 = \sqrt{(25/49)}$

⇒ $r_1/r_2 = 5/7$

Therefore, ratio of their radii is 5 : 7.

6. *The diameter of a wheel of a motorcycle is 63 cm. How many revolutions will it make to travel 99 km?*

Solution:

The diameter of the wheel of a motorcycle = 63 cm

Therefore, circumference of the wheel of motorcycle = πd

= 22/7 × 63

= 198 cm

Total distance travelled by motorcycle = 99 km

= 99 × 1000

= 99 × 1000 × 100 cm

Therefore, number of revolutions = (99 × 1000 × 100)/198 = 50000

7. *The diameter of a wheel of cycle is 21 cm. It moves slowly along a road. How far will it go in 500 revolutions?*

Solution:

In revolution, distance that wheel covers = circumference of wheel Diameter of wheel = 21 cm

Therefore, circumference of wheel = πd

= 22/7 × 21

= 66 cm

So, in 1 revolution distance covered = 66 cm

In 500 revolution distance covered = 66 × 500 cm

= 33000 cm

= 33000/100 m

= 330 m

8. *The circumference of a circle exceeds the diameter by 20 cm. Find the radius of the circle.*

Solution:

Let the radius of circle of = r m.

Then circumference = 2 πr

Since, circumference exceeds diameter by 20

Therefore, according to question;

2 πr = d + 20

⇒ 2 πr = 2r + 20

⇒ 2 × (22/7) × r = 2r + 20

⇒ 44r/7 - 2r = 20

⇒ (44r - 14r)/7 = 20

⇒ 30r/7 = 20

⇒ r = (7 × 20)/30

⇒ r =14/3

So, the radius of circle = 14/3 cm = 42/3 cm

9. *A piece of wire in the form of rectangle 40 cm long and 26 cm wide is again bent to form a circle. Find the radius of the circle.*

Solution:

Length of wire = Perimeter of rectangle

= 2(l + b)

= 2(40 + 26)

= 2 × 66

= 132 cm

When it is again bent to form a circle, then

Perimeter of circle = Perimeter of rectangle

$2\pi r = 132$ cm

$\Rightarrow 2 \times 22/7 \times r = 132$

$\Rightarrow r = (132 \times 7)/(2 \times 22)$

$\Rightarrow r = 21$ cm

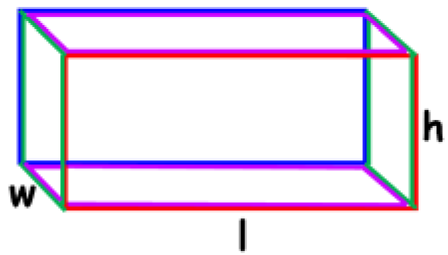
SURFACE AREA AND VOLUMES OF PRISMS AND PYRAMIDS

Both prism and pyramid are basically 3D shapes. Even though we have different formulas to find surface area of prism and pyramid, the basic idea of finding surface area is to add the areas of all the faces.

First, let us look at, how to find surface area of a prism.

Surface Area of Prism

Let us consider the rectangle prism given below.



Here is the basic idea to find surface area of the above rectangular prism.

Surface Area = Sum of areas of all six faces

Let us find the area of each face separately.

Area of the front face (red colored) = l x h

Area of the back face (blue colored) = l x h

Area of the left side face (green colored) = w x h

Area of the right side face (green colored) = w x h

Area of the top portion (purple colored) = l x w

Area of the base (purple colored) = l x w

Now,

Surface area = lh + lh + wh + wh + lw + lw

Surface area = 2lh + 2wh + 2lw

Surface area = 2(lh + wh + lw)

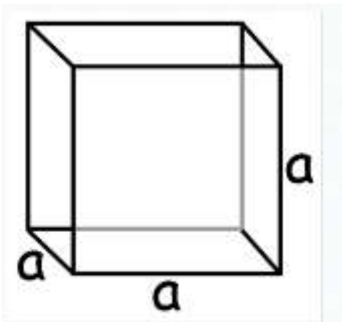
This is the formula to find surface area of a rectangular prism.

Note:

Rectangular prism is also known as cuboid.

We can apply the above explained basic idea to find surface area of any prism without remembering the formulas.

Let us find surface area of the cube given below.



We know that the shape of each face of a cube is a square.

In the above cube, the side length of each face is "a".

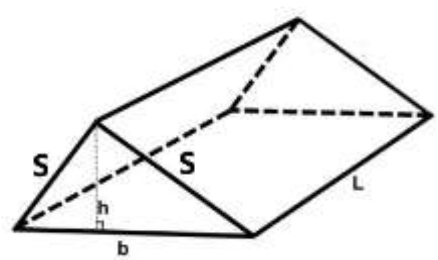
So, area of each face (square) = a x a = a^2

Therefore,

Surface area of cube = 6 x area of each face

Surface area of cube = $6a^2$

Now, let us find surface area of the triangular prism given below.



In the above triangular prism, there are five faces. The shape of the base and the two slanting faces is rectangle. The shape of two faces on the left side and right side is triangle.

For the given triangular prism,

Area of the base = Lb.

Area of the first slanting face = Ls

Area of the other slanting face = Ls

Area of the front face = (1/2)bh

Area of the back face = (1/2)bh

So,

surface area = sum of the area of 5 faces

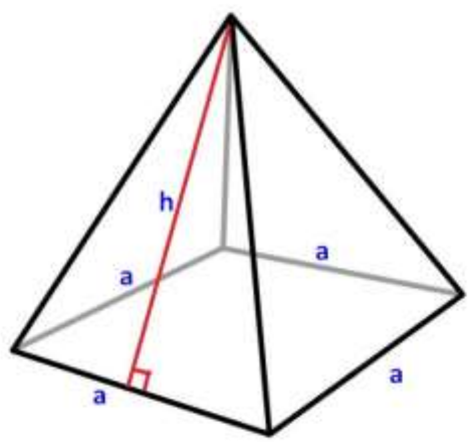
surface area = Lb + 2Ls + 2 x (1/2)bh

Surface area of triangular prism = Lb+2Ls+bh

Now let us look at, how to find surface area of a prism.

Surface Area of Pyramid

Let us consider the pyramid with square base given below.



Here is the basic idea to find surface area of the above pyramid.

Surface Area = Sum of areas of all five faces (Including the base)

For any pyramid, if the shape of the base is square, then we will have four side walls. The shape of each side wall will be a triangle with equal area.

In the above pyramid, the base is a square with side length "a" and each wall is a triangle with base "a" and height "h"

Let us find the area of each face separately.

Area of the base = a x a = a^2

Area of each side wall = (1/2) ah

Area of all four side walls = 4 x (1/2) ah = 2ah

Surface area of the above pyramid is

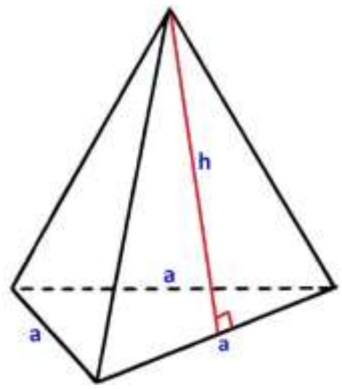
= a^2 + 2ah

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

This is the formula to find surface area of a pyramid with square base.

We can apply the above explained basic idea to find surface area of a pyramid with triangular base.

Let us find surface area of a pyramid with triangular base.



For any pyramid, if the shape of the base is equilateral triangle, then we will have three side walls. The shape of each side wall will be a triangle with equal area.

In the above pyramid, the base is an equilateral triangle with side length "a".

And each wall is a triangle with base "a" and height "h"

Let us find the area of each face separately.

Area of the base = $(\sqrt{3}/4)a^2$

Area of each side wall = (1/2)ah

Area of all 3 side walls = 3 x (1/2)ah = (3/2)ah

Surface area of the above pyramid is

= $(\sqrt{3}/4)a^2 + (3/2)ah$

This is the formula to find surface area of a pyramid with equilateral triangle base.

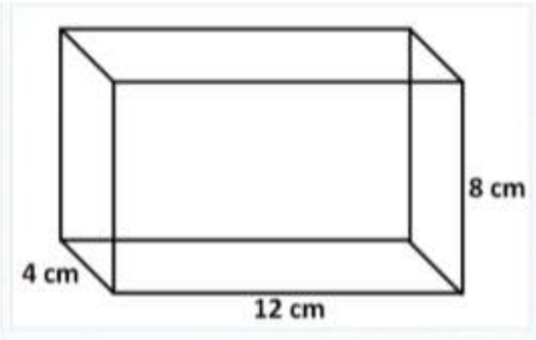
Note:

If the base is not equilateral triangle and it is either scalene triangle or isosceles triangle, then the area of side walls will not be equal. We have to find area of each side wall separately.

Practice Problems

Problem 1:

Find the surface area of the cuboid shown below.



Solution:

Surface area of cuboid is

= Sum of areas of all six faces

In cuboid, each face is a rectangle. So we can use area of rectangle formula to get area of each face.

Area of the front face = 8 x 12 = 96 cm^2

Area of the back face = 8 x 12 = 96 cm^2

Area of the left side face = 4 x 8 = 32 cm^2

Area of the right side face = 4 x 8 = 32 cm^2

Area of the top portion = 4 x 12 = 48 cm^2

Area of the base = 4 x 12 = 48 cm^2

Surface area of the above cuboid is

= Sum of areas of all six faces

= 96 + 96 + 32 + 32 + 48 + 48

= 96 + 96 + 32 + 32 + 48 + 48

= 352 cm^2

Alternative Method:

We can use the formula given below to find surface area of cuboid.

Formula for surface area of cuboid is

= 2(lh + wh + lw)

Substitute l = 12, w = 4 and h = 8.

= 2(12x8 + 4x8 + 12x4)

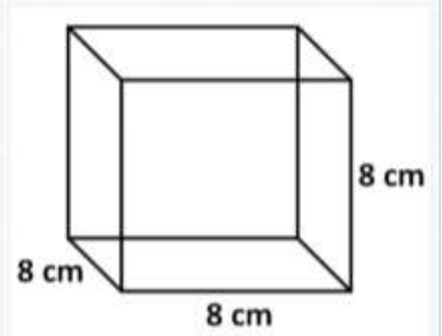
= 2(96 + 32 + 48)

= 2(176)

= 352 cm^2

Problem 2:

Find the surface area of the cube shown below.



Solution:

We know that the shape of each face of a cube is a square.

In the above cube, the side length of each face is "8".

So, area of each face (square) is

= 8 x 8

= 64 cm^2

Therefore, surface area of the cube is

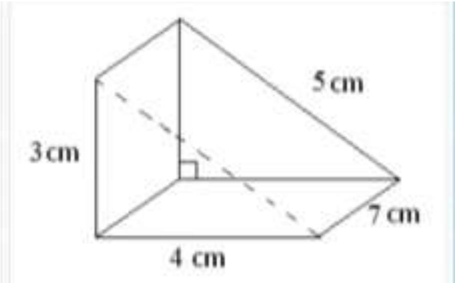
= 6 x area of each face

= 6 x 64

= 384 sq.cm

Problem 3:

Find the surface area of the triangular prism shown below.



Solution:

In the above triangular prism, there are five faces. The shape of the base, vertical face and slanting face is rectangle. The shape of two faces on the left side and right side is triangle.

For the given triangular prism,

Area of the base = 7 x 4 = 28 cm^2

Area of the vertical face = 3 x 7 = 21 cm^2

Area of the slanting face = 5 x 7 = 35 cm^2

Area of the front face = (1/2) x 4 x 3 = 6 cm^2

Area of the back face = (1/2) x 4 x 3 = 6 cm^2

So, surface area of the above triangular prism is

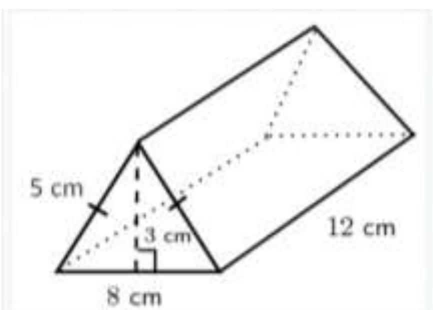
= sum of the area of 5 faces

= 28 + 21 + 35 + 6 + 6

= 96 cm^2

Problem 4:

Find e surface area of the triangular prism shown below.



Solution:

In the above triangular prism, there are five faces. The shape of the base and the two slanting faces is rectangle. The shape of two faces on the left side and right side is triangle.

For the given triangular prism,

Area of the base = 8 x 12 = 96 cm^2

Area of the first slanting face = 12 x 5 = 60 cm^2

Area of the other slanting face = 12 x 5 = 60 cm^2

Area of the front face = (1/2) x 8 x 3 = 12 cm^2

Area of the back face = (1/2) x 8 x 3 = 12 cm^2

So, surface area of the above the triangular prism is

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

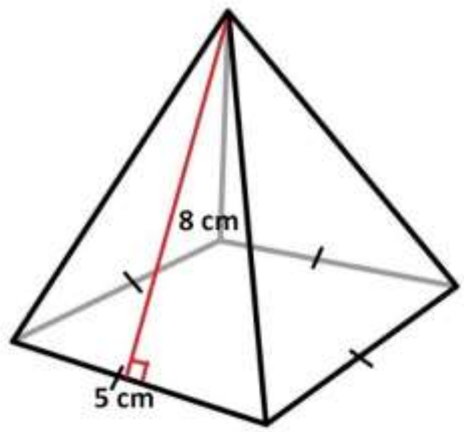
= sum of the area of 5 faces

= 96 + 60 + 60 + 12 + 12

= 240 cm^2

Problem 5:

Find the surface area of the pyramid shown below.



Solution:

Surface area of the pyramid is

= Sum of areas of all 5 faces

In the above pyramid, the base is a square with side length 5 cm and each wall is a triangle with base 5 cm and height 8 cm.

Let us find the area of each face separately.

Area of the base = 5 x 5 = 25 sq.cm

Area of each side wall = (1/2) x 5 x 8 = 20 sq.cm

Area of all 4 side walls = 4 x 20 = 80 sq.cm

Surface area of the above pyramid is

= 25 + 80

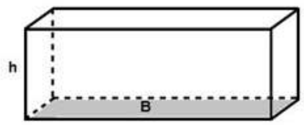
= 105 sq.cm

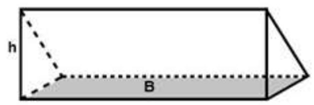
The surface area is the area that describes the material that will be used to cover a geometric solid. When we determine the surface areas of a geometric solid we take the sum of the area for each geometric form within the solid.

The volume is a measure of how much a figure can hold and is measured in cubic units. The volume tells us something about the capacity of a figure.

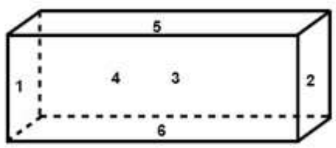
CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

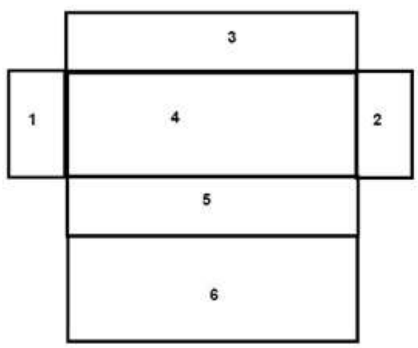
A prism is a solid figure that has two parallel congruent sides that are called bases that are connected by the lateral faces that are parallelograms. There are both rectangular and triangular prisms.

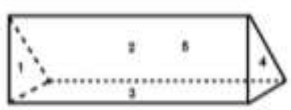


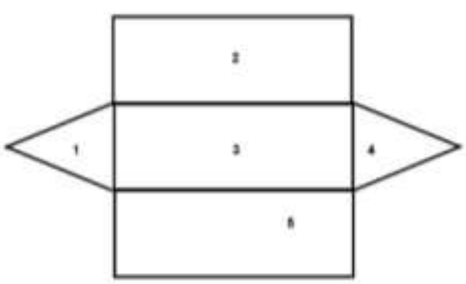


To find the surface area of a prism (or any other geometric solid) we open the solid like a carton box and flatten it out to find all included geometric forms.







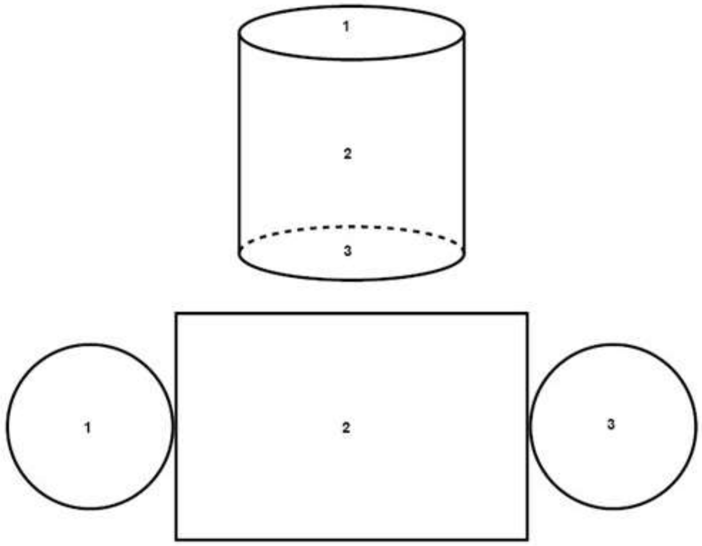


To find the volume of a prism (it doesn't matter if it is rectangular or triangular) we multiply the area of the base, called the base area B, by the height h.

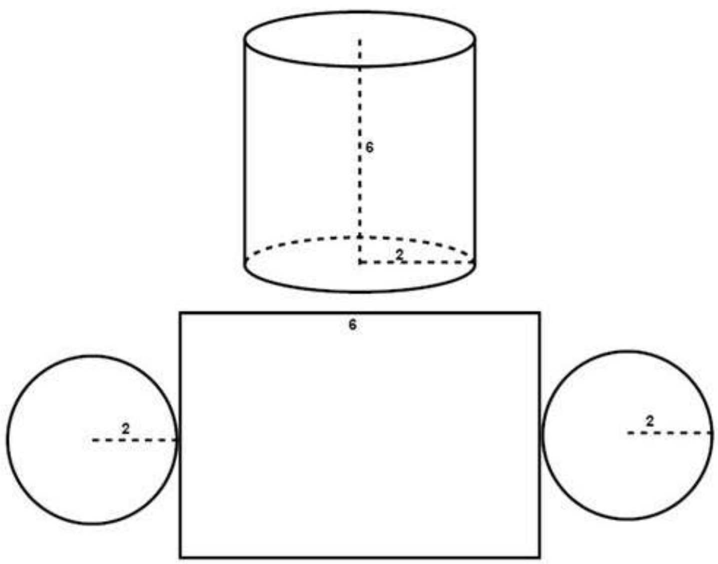
$V=B\cdot hV=B\cdot h$

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

A cylinder is a tube and is composed of two parallel congruent circles and a rectangle which base is the circumference of the circle.



Example



The area of one circle is:

A=πr2A=πr2

A=π·22A=π·22

A=π·4A=π·4

A≈12.6A≈12.6

The circumference of a circle:

$C=\pi d C=\pi d$

$C=\pi \cdot 4 C=\pi \cdot 4$

$C\approx 12.6 C\approx 12.6$

The area of the rectangle:

$A=C\cdot h A=C\cdot h$

$A=12.6\cdot 6 A=12.6\cdot 6$

$A\approx 75.6 A\approx 75.6$

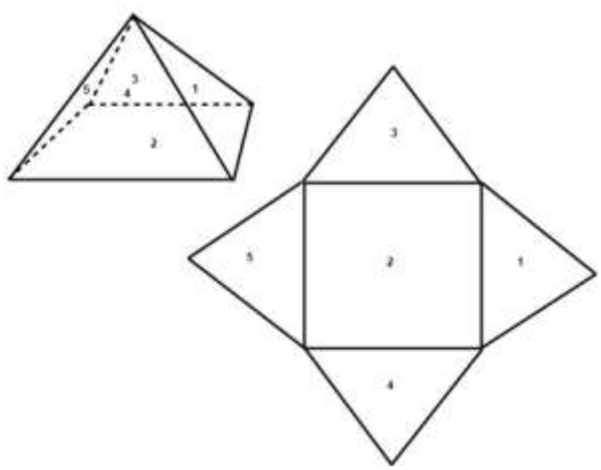
The surface area of the whole cylinder:

$A=75.6+12.6+12.6=100.8 units2 A=75.6+12.6+12.6=100.8 units2$

To find the volume of a cylinder we multiply the base area (which is a circle) and the height h.

$V=\pi r2\cdot h V=\pi r2\cdot h$

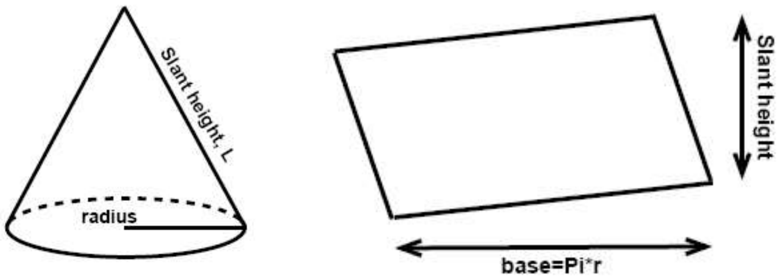
A pyramid consists of three or four triangular lateral surfaces and a three or four sided surface, respectively, at its base. When we calculate the surface area of the pyramid below we take the sum of the areas of the 4 triangles area and the base square. The height of a triangle within a pyramid is called the slant height.



The volume of a pyramid is one third of the volume of a prism.

$V=13\cdot B\cdot h V=13\cdot B\cdot h$

The base of a cone is a circle and that is easy to see. The lateral surface of a cone is a parallelogram with a base that is half the circumference of the cone and with the slant height as the height. This can be a little bit trickier to see, but if you cut the lateral surface of the cone into sections and lay them next to each other it's easily seen.

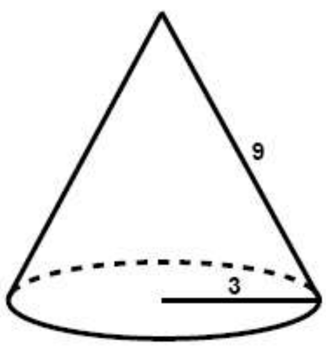


The surface area of a cone is thus the sum of the areas of the base and the lateral surface:

Abase=πr2andALS=πrlAbase=πr2andALS=πrl

A=πr2+πrlA=πr2+πrl

Example



Abase=πr2Abase=π·32Abase≈28.3andALS=πrlALS=π·3·9ALS≈84.8Abase=πr2andALS=πrlAbase=π·32ALS=π·3·9Abase≈28.3ALS≈84.8

A=πr2+πrl=28.3+84.8=113.1units2A=πr2+πrl=28.3+84.8=113.1units2

The volume of a cone is one third of the volume of a cylinder.

V=13π·r2·hV=13π·r2·h

Example

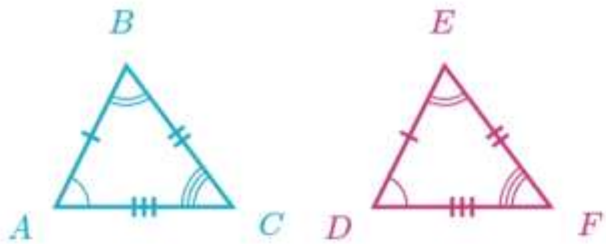
Find the volume of a prism that has the base 5 and the height 3.

B=3·5=15B=3·5=15

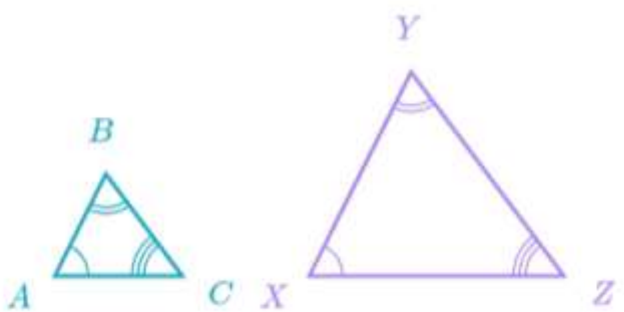
V=15·3=45units3

CONGRUENCE, SYMMETRY AND OTHER PROPERTIES OF GIVEN SHAPES.

Congruent triangles have both the same shape and the same size. In the figure below, triangles ABC and DEF are congruent; they have the same angle measures and the same side lengths.



Similar triangles have the same shape, but not necessarily the same size. In the figure below, triangles ABC and XYZ are similar, but not congruent; they have the same angle measures, but not the same side lengths.



Note: If two objects are congruent, then they are also similar.

What skills are tested?

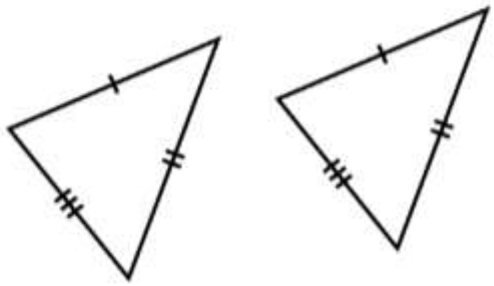
- Determining whether two triangles are congruent
- Determining whether two triangles are similar
- Using similarity to find a missing side length

WHAT ARE THE TRIANGLE CONGRUENCE CRITERIA?

Two triangles are congruent if they meet one of the following criteria.

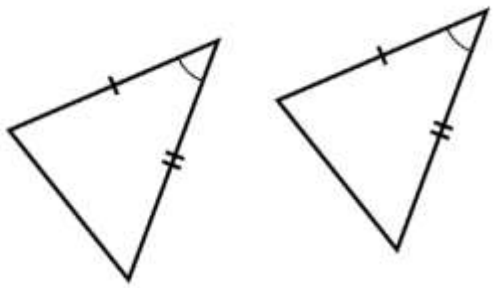
SSS

: All three pairs of corresponding sides are equal.



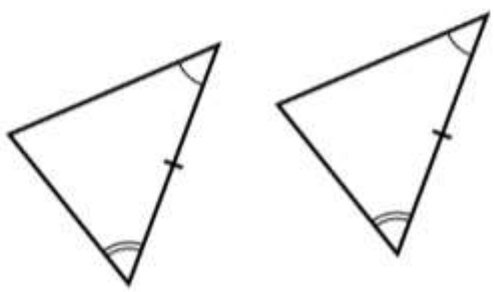
SAS

: Two pairs of corresponding sides and the corresponding angles between them are equal.



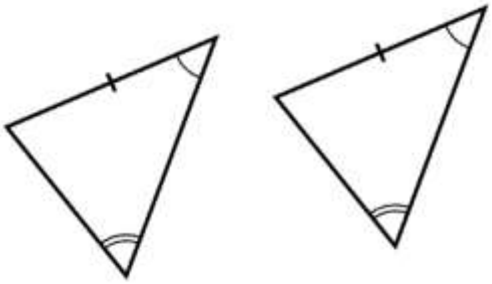
ASA

: Two pairs of corresponding angles and the corresponding sides between them are equal.



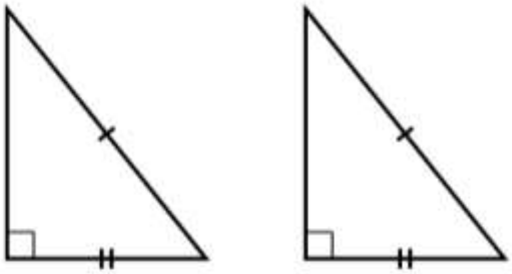
AAS

: Two pairs of corresponding angles and one pair of corresponding sides (not between the angles) are equal.



HL

: The pair of hypotenuses and another pair of corresponding sides are equal in two right triangles.

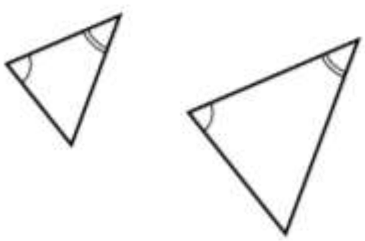


WHAT ARE THE TRIANGLE SIMILARITY CRITERIA?

Two triangles are similar if they meet one of the following criteria.

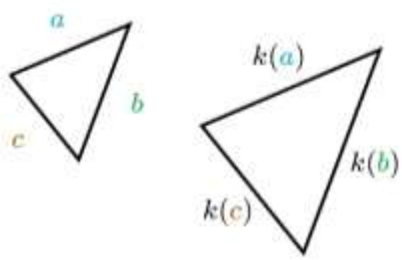
AA

: Two pairs of corresponding angles are equal.



SSS

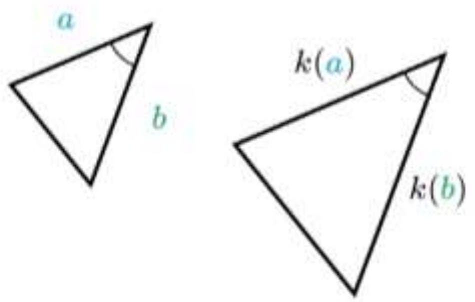
: Three pairs of corresponding sides are proportional.



CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

SAS

: Two pairs of corresponding sides are proportional and the corresponding angles between them are equal.



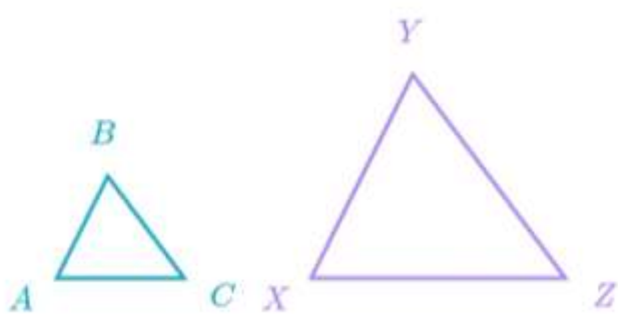
FINDING MISSING SIDE LENGTHS IN SIMILAR TRIANGLES

The SSS similarity criterion allows us to calculate missing side lengths in similar triangles. For similar triangles ABC*ABC*A, B, C and XYZ*XYZ*X, Y, Z shown below:

$$YZ = k(BC)$$

$$XZ = k(AC)$$

$$\frac{XY}{AB} = \frac{YZ}{BC} = \frac{XZ}{AC} = k$$



TO CALCULATE A MISSING SIDE LENGTH, WE:

1. Write a proportional relationship using two pairs of corresponding sides.
2. Plug in known side lengths. We need to know 333 of the 444 side lengths to solve for the missing side length.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

3. Solve for the missing side length.

THINGS TO REMEMBER

Congruent triangles have the same corresponding angle measures and side lengths. The triangle congruence criteria are:

- SSS (Side-Side-Side)
- SAS (Side-Angle-Side)
- ASA (Angle-Side-Angle)
- AAS (Angle-Angle-Side)
- HL (Hypotenuse-Leg, right triangle only)

Similar triangles have the same corresponding angle measures and proportional side lengths. The triangle similarity criteria are:

- AA (Angle-Angle)
- SSS (Side-Side-Side)
- SAS (Side-Angle-Side)

If triangles ABC and XYZ are similar, then their corresponding side lengths have the same ratio:

$$\frac{XY}{AB} = \frac{YZ}{BC} = \frac{XZ}{AC} = k$$

UNIT 2
CONSTRUCTION

CONSTRUCTIONS

Here you will learn to create formal geometric constructions by hand and with dynamic geometry software. You will learn to both copy and bisect segments and angles. You will use your knowledge of angles to construct parallel and perpendicular lines. You will use these basic constructions together with your knowledge of the properties of regular polygons to construct regular polygons.

You learned that a drawing is a rough sketch while a construction is a step-by-step process for creating an accurate geometric figure. A compass and straightedge or folded paper can help when performing constructions by hand.

You mastered copying and bisecting line segments and angles. You used your knowledge of corresponding angles to copy an angle in order to create parallel lines. You learned how to construct an equilateral triangle, a square, and a regular hexagon inscribed in a circle by hand.

Finally, you learned how to use dynamic geometry software such as *Geogebra* to create a specific polygon using the properties of that polygon.

CONSTRUCTING PERPENDICULAR PARALLEL LINES

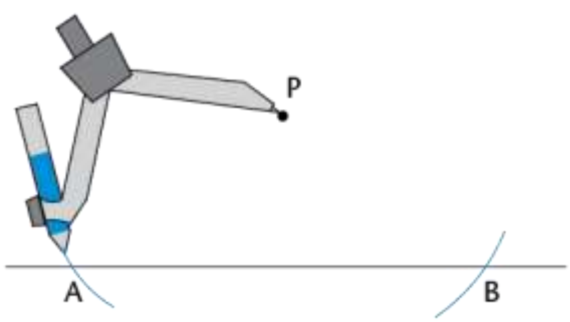
CONSTRUCTING PERPENDICULAR LINES

A perpendicular line from a given point

1. Read through the following steps.

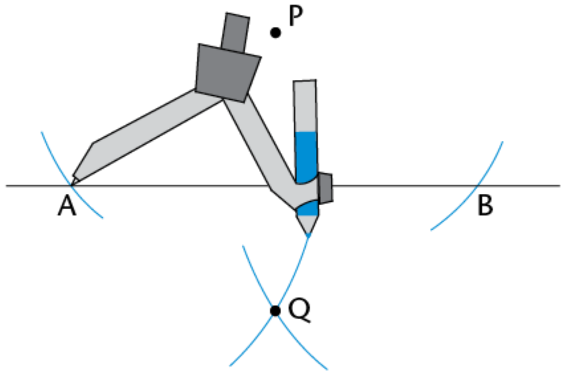
Step 1

Place your compass on the given point (point P). Draw an arc across the line on each side of the given point. Do not adjust the compass width when drawing the second arc.



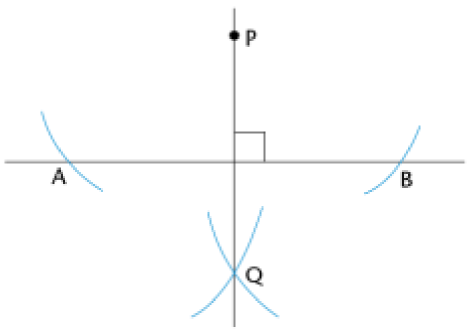
Step 2

From each arc on the line, draw another arc on the opposite side of the line from the given point (P). The two new arcs will intersect.



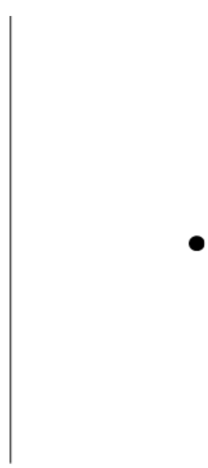
Step 3

Use your ruler to join the given point (P) to the point where the arcs intersect (Q).



PQ is perpendicular to AB. We also write it like this: PQ âŠ¥ AB.

2. Use your compass and ruler to draw a perpendicular line from each given point to the line segment:

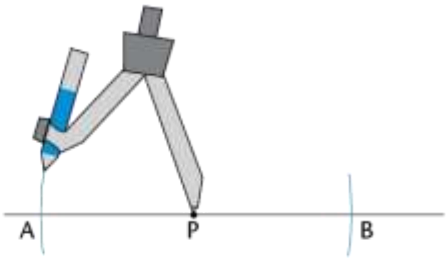


A perpendicular line at a given point on a line

1. Read through the following steps.

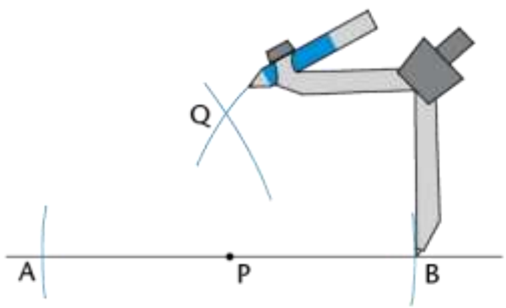
Step 1

Place your compass on the given point (P). Draw an arc across the line on each side of the given point. Do not adjust the compass width when drawing the second arc.



Step 2

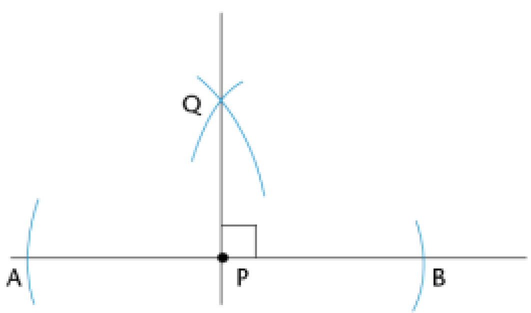
Open your compass so that it is wider than the distance from one of the arcs to the point P. Place the compass on each arc and draw an arc above or below the point P. The two new arcs will intersect.



Step 3

Use your ruler to join the given point (P) and the point where the arcs intersect (Q).

PQ âŠ¥ AB



Use your compass and ruler to draw a perpendicular at the given point on each line

COPYING AND BISECTING ANGLES AND LINES

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

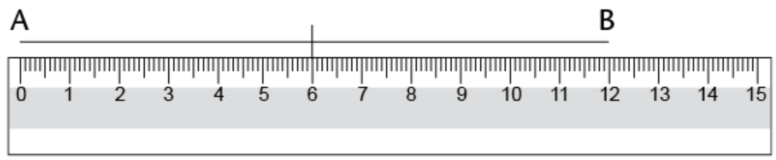
BISECTING LINES

When we construct, or draw, geometric figures, we often need to bisect lines or angles. **Bisect** means to cut something into two equal parts. There are different ways to bisect a line segment.

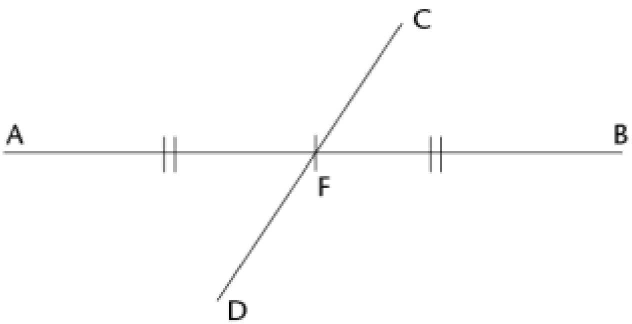
BISECTING A LINE SEGMENT WITH A RULER

1. Read through the following steps.

Step 1: Draw line segment AB and determine its midpoint.



Step 2: Draw any line segment through the midpoint.



The small marks on AF and FB show that AF and FB are equal.

CD is called a **bisector** because it bisects AB. AF = FB.

2. Use a ruler to draw and bisect the following line segments: AB = 6 cm and XY = 7 cm.

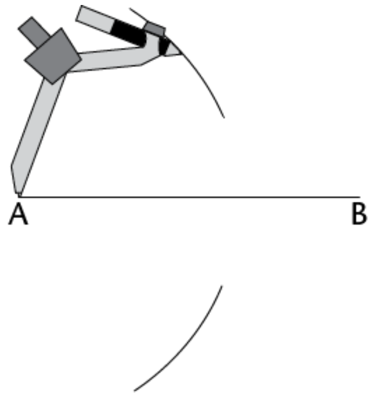
In Grade 6, you learnt how to use a compass to draw circles, and parts of circles called arcs. We can use arcs to bisect a line segment.

BISECTING A LINE SEGMENT WITH A COMPASS AND RULER

1. Read through the following steps.

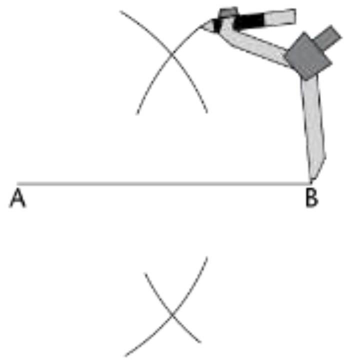
Step 1

Place the compass on one endpoint of the line segment (point A). Draw an arc above and below the line. (Notice that all the points on the arc aboveand below the line are the same distance from point A.)



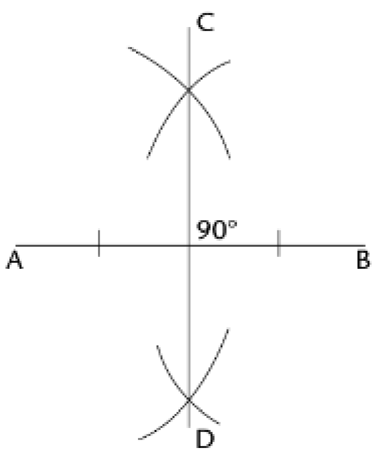
Step 2

Without changing the compass width, place the compass on point B. Draw an arc above and below the line so that the arcs cross the first two. (The two points where the arcs cross are the same distance away from point A and from point B.)



Step 3

Use a ruler to join the points where the arcs **intersect**. This line segment (CD) is the bisector of AB.



Intersect means to cross or meet.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

A **perpendicular** is a line that meets another line at an angle of 90°.

Notice that CD is also **perpendicular** to AB. So it is also called a **perpendicular bisector**.

2. Work in your exercise book. Use a compass and a ruler to practise drawing perpendicular bisectors on line segments.

Try this!

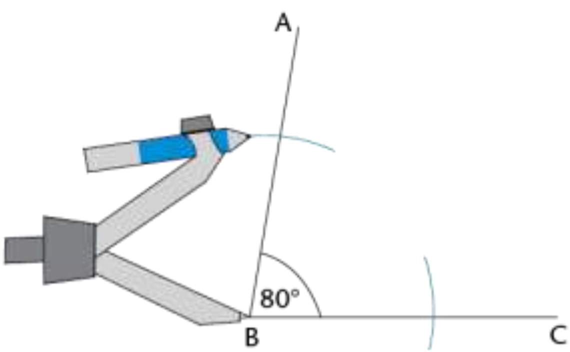
Work in your exercise book. Use only a protractor and ruler to draw a perpendicular bisector on a line segment. (Remember that we use a protractor to measure angles.)

BISECTING ANGLES

1. Read through the following steps.

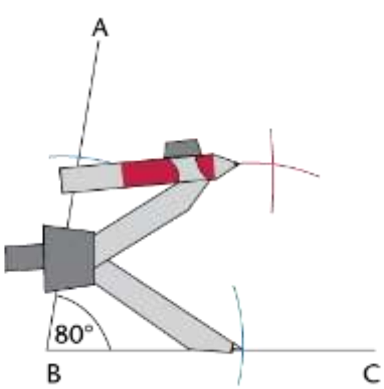
Step 1

Place the compass on the vertex of the angle (point B). Draw an arc across each arm of the angle.



Step 2

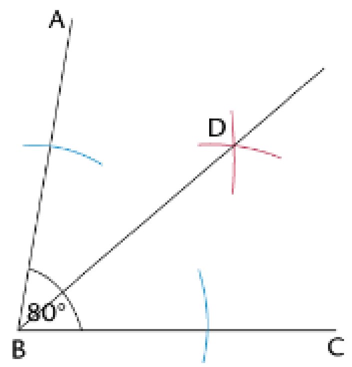
Place the compass on the point where one arc crosses an arm and draw an arc inside the angle. Without changing the compass width, repeat for the other arm so that the two arcs cross.



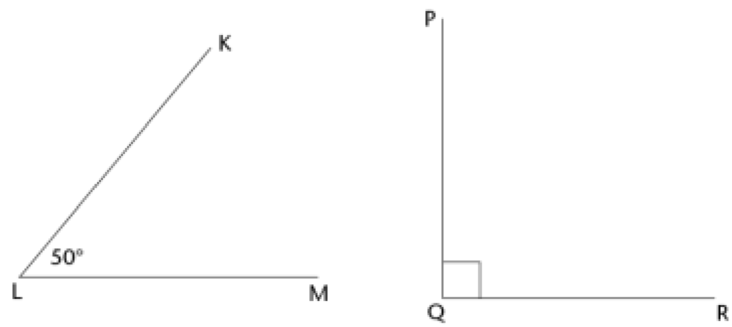
Step 3

Use a ruler to join the vertex to the point where the arcs intersect (D).

DB is the bisector of $\hat{ABC}$.



2. Use your compass and ruler to bisect the angles below.



You could measure each of the angles with a protractor to check if you have bisected the given angle correctly.

COPYING LINES AND ANGLES

1. **Compass:** A device that allows you to create a circle with a given radius. Not only can compasses help you to create circles, but also they can help you to copy distances.
2. **Straightedge:** Anything that allows you to produce a straight line. A straightedge should not be able to measure distances. An index card works well as a straightedge. You can also use a ruler as a straightedge, as long as you only use it to draw straight lines and not to measure.
3. **Paper:** When a geometric figure is on a piece of paper, the paper itself can be folded in order to construct new lines.

Let's take a look at some example problems.

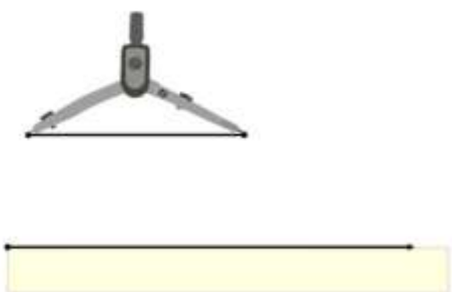
1. Use a straightedge to draw a line segment on your paper like the one shown below. Then, use your straightedge and compass to copy the line segment exactly.

[Figure 2]

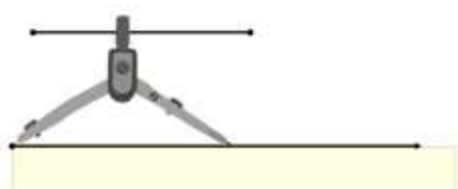
First use your straightedge and pencil to create a new ray.

[Figure 3]

Now, you have one endpoint of your line segment. Your job is to figure out where the other endpoint should go on the ray. Use your compass to measure the width of the original line segment.

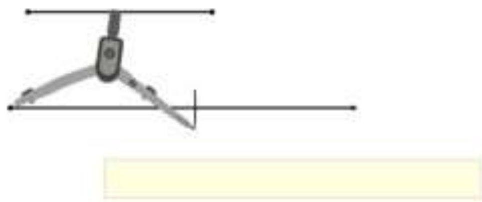


Now, move the compass so that the tip is on the endpoint of the ray.



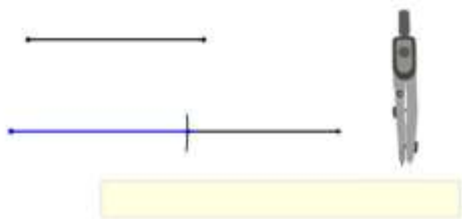
[Figure 5]

You can now see where the endpoint of the segment should lie on the ray. Draw a little arc with the compass to mark where the endpoint should go.



[Figure 6]

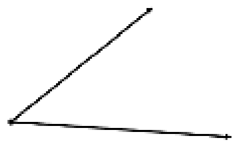
You can use your straightedge to draw the copied line segment in a different color if you wish.



[Figure 7]

Note that in this construction, the compass was used to copy a distance. This is one of the primary uses of a compass in constructions.

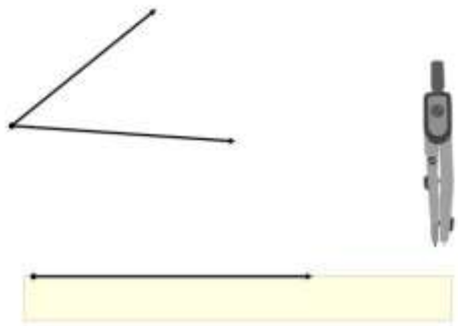
2. Use a straightedge to draw an angle on your paper like the one shown below. Then, use your straightedge and compass to copy the angle exactly.



[Figure 8]

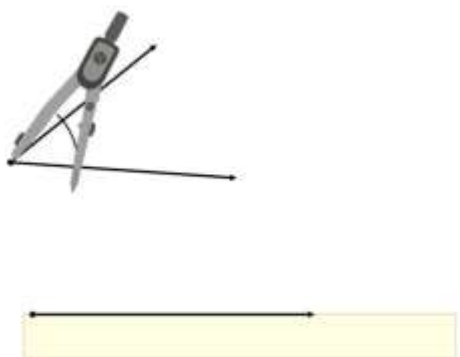
Keep in mind that what defines the angle is the opening between the two rays. The lengths of the rays are not relevant.

Start by using your straightedge and pencil to draw a new ray. This will be the bottom of the two rays used to create the angle.



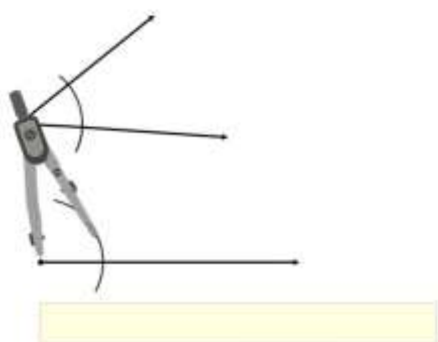
[Figure 9]

Next, use your compass to make an arc through the original angle. It does not matter how wide you open your compass for this.



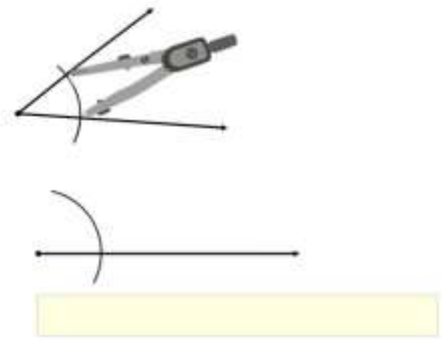
[Figure 10]

Next, leave your compass open to the same width, and make a similar arc through the new ray.



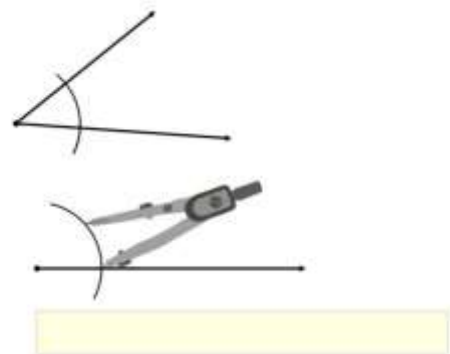
[Figure 11]

3. Now, you know that the second ray necessary to create the new angle will go somewhere through that arc. Measure the width of the arc on the original angle using the compass.



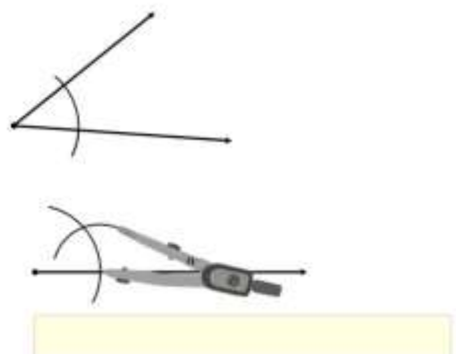
[Figure 12]

Leave the compass open to the same width, and move it to the new angle.



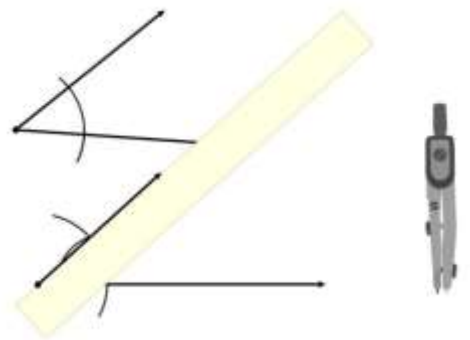
[Figure 13]

Make a mark to show where the pencil on the compass intersects the arc.



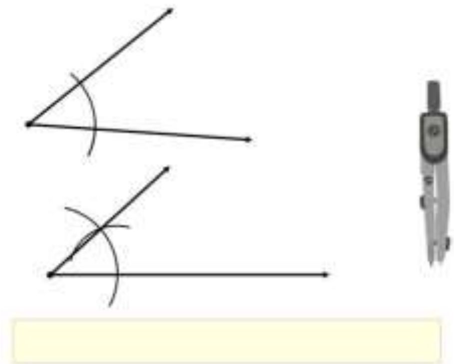
[Figure 14]

4. Use a straightedge to draw another ray that passes through the point of intersection of the two compass markings.



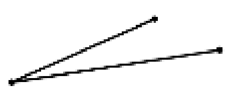
[Figure 15]

You have now copied the angle exactly.



[Figure 16]

5. An angle is created from two line segments. Use a straightedge to draw a similar figure on your paper. Then, use the straightedge and compass to copy the figure exactly.



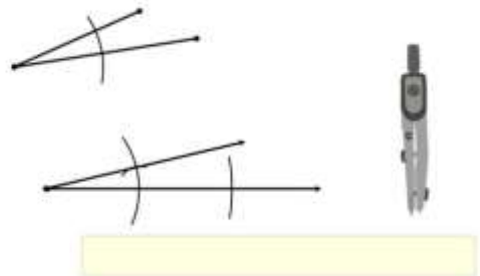
[Figure 17]

To copy this figure, you will need to copy both the line segments and the angle.

Start by copying the line segment on the bottom using the process outlined in #1 (draw a ray, use the compass to measure the width of the line segment, mark off the endpoint on the ray).

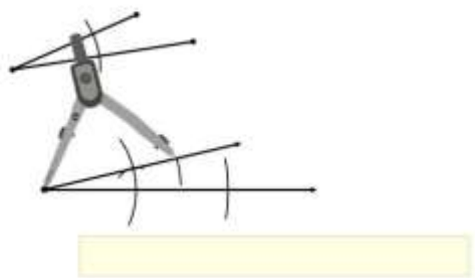
[Figure 18]

Next, copy the angle using the process outlined in #2 (draw an arc through the angle and draw the same arc through the new ray, measure the width of the arc, draw a new ray through the intersection of the two markings).



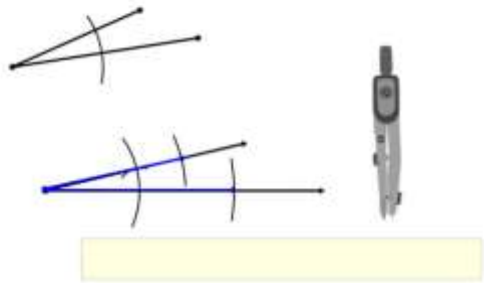
[Figure 19]

Finally, copy the second line segment by measuring its length using the compass and marking off the correct spot for the endpoint.



[Figure 20]

You can now draw the copied segments in a different color for emphasis.



[Figure 21]

Examples

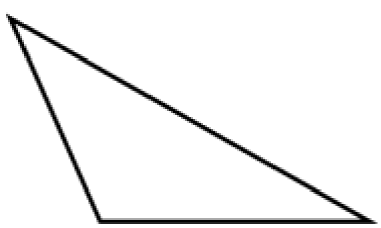
Example 1

Earlier, you were asked to describe two ways how to use a compass and a straightedge to copy a triangle.

To copy a triangle means to create a congruent triangle. There are four triangle congruence criteria that work for any type of triangle: SSS, SAS, AAS, ASA. You can use SSS, SAS, or ASA combined with copying angles and line segments to copy a triangle.

1. SSS: Copy one-line segment. Copy the other two line segments so that their endpoints intersect. *This construction will be explored in Guided Practice #1.*
2. SAS: Copy one line segment. Copy an angle from one of the endpoints of the line segment. Copy a second line segment onto the ray created by the copied angle. Connect the endpoints to form the triangle. *This is very similar to Example C. This construction is explored in Guided Practice #2.*
3. ASA: Copy one line segment. Copy two angles, one from each endpoint. The intersection of the angles will produce the third vertex of the triangle. *This construction is explored in Guided Practice #3.*

You drew a triangle similar to the one below for the concept problem.

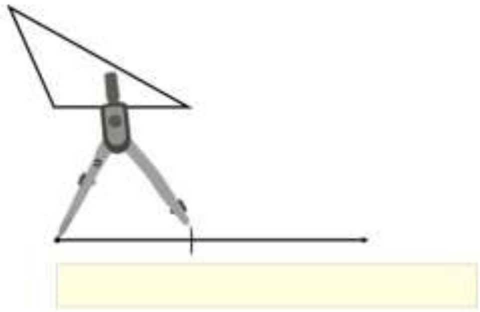


[Figure 22]

Example 2

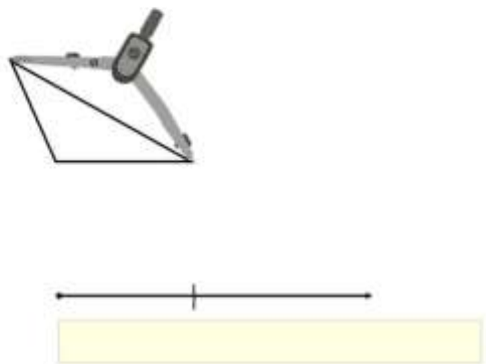
Copy your triangle using SSS.

Start by copying one line segment. Here, the base line segment is copied.



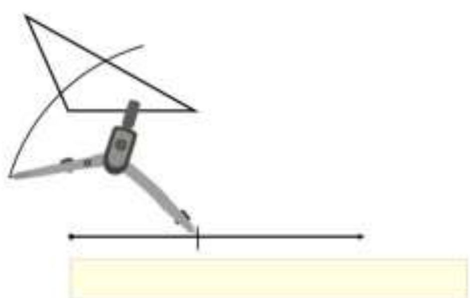
[Figure 23]

Next, use the compass to measure the length of one of the other sides of the triangle.



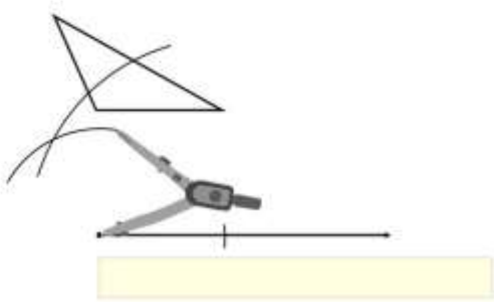
[Figure 24]

Move the compass to the location of the new triangle and make an arc to mark the length of the second side of the triangle from the correct endpoint.



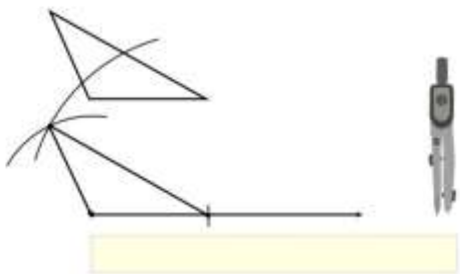
[Figure 25]

Repeat with the third side of the triangle.



[Figure 26]

The point where the arcs intersect is the third vertex of the triangle. Connect to form the triangle.



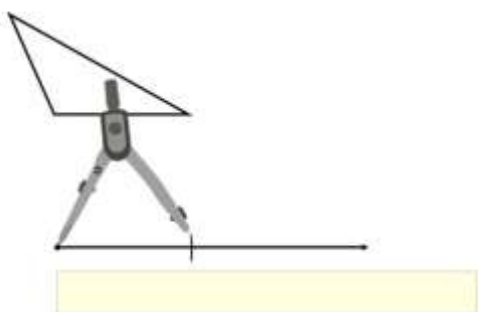
[Figure 27]

Note that with this method, you have only used the lengths of the sides of the triangle (as opposed to any angles) to construct the new triangle.

Example 3

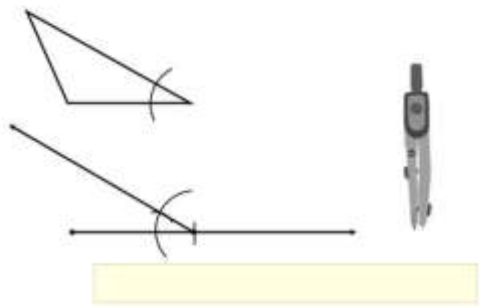
Copy your triangle using SAS.

Start by copying one line segment. Here, the base of the triangle is copied.



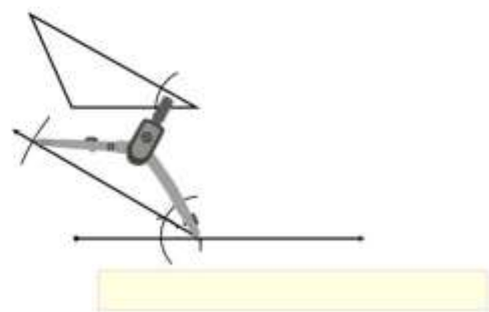
[Figure 28]

Next, copy the angle at one of the endpoints of the line segment.



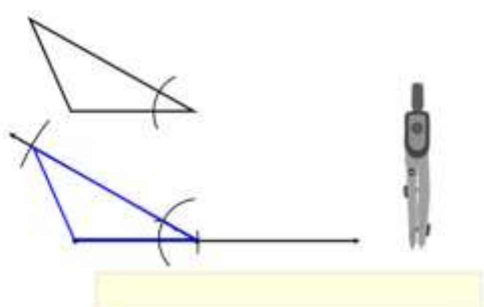
[Figure 29]

Copy the second side of the triangle (that creates the angle you copied) onto the ray that you just drew.



[Figure 30]

Connect to form the triangle.

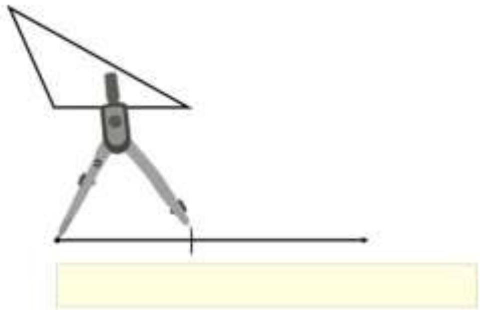


[Figure 31]

Example 4

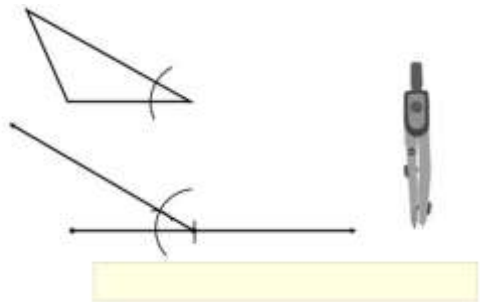
Copy your triangle using ASA.

Start by copying one line segment.



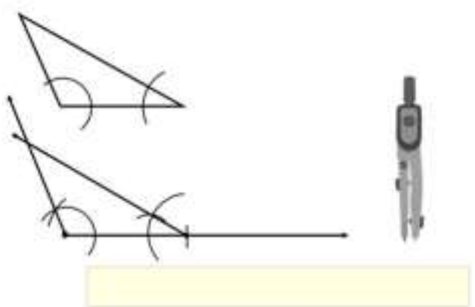
[Figure 32]

Next, copy the angle at one of the endpoints.



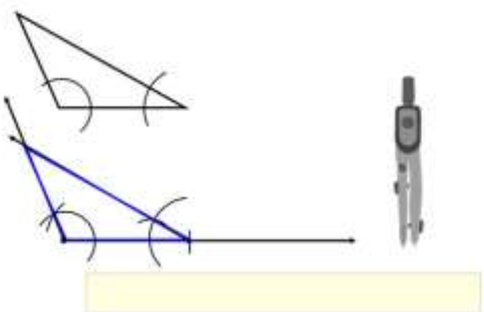
[Figure 33]

Copy the angle at the other endpoint of the line segment.



[Figure 34]

Connect to form the triangle.



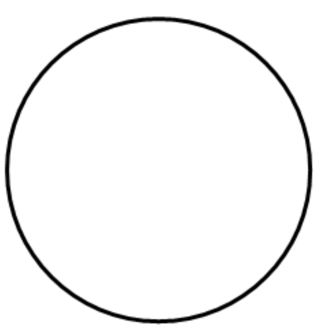
CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

[Figure 35]

Review

1. What is the difference between a drawing and a construction?
2. What is the difference between a straightedge and a ruler?
3. Describe the steps for copying a line segment.
4. Describe the steps for copying an angle.
5. When copying an angle, do the lengths of the lines matter? Explain.
6. Explain the connections between copying a triangle and the triangle congruence criteria.
7. Draw a line segment and copy it with a compass and straightedge.
8. Draw another line segment and copy it with a compass and straightedge.
9. Draw an angle and copy it with a compass and straightedge.
10. Draw another angle and copy it with a compass and straightedge.
11. Use your straightedge to draw a triangle. Copy the triangle using SSS≅. Describe your steps.
12. Copy the triangle from #11 using SAS≅. Describe your steps.
13. Copy the triangle from #11 using ASA≅. Describe your steps.
14. Can you copy the triangle from #11 using AAS≅? Explain.
15. Compare the methods for copying the triangle. Is one method easier than the others? Explain.

Use your compass to construct a circle like the one shown below on a piece of paper. Describe how to fold the paper two times in order to help you construct a square.



[Figure 1]

CONSTRUCTING ANGLES

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

In construction of angles by using compass we will learn how to construct different angles with the help of ruler and compass.

1. Construction of an Angle of 60° by using Compass

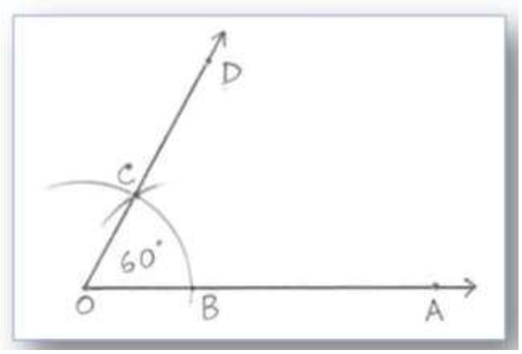
Step of Construction:

(i) *Draw a ray OA.*

(ii) *With O as centre and any suitable radius draw an arc above OA cutting it at a point B.*

(iii) *With B as centre and the same radius as before, draw another arc to cut the previous arc at C.*

(iv) *Join OC and produce it to D.*



Then **∠AOD = 60°**.

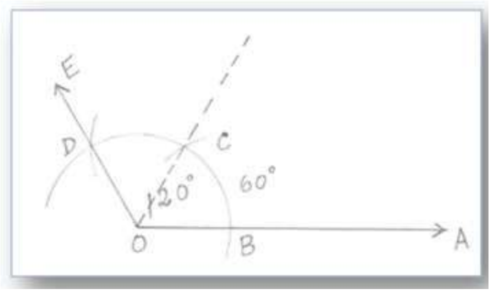
2. Construction of an Angle of 120° by using Compass

Step of Construction:

(i) *Draw a ray OA.*

(ii) *With O as centre and any suitable radius draw an arc cutting OA at B.*

(iii) *With B as centre and the same radius cut the arc at C, then with C as centre and same radius cut the arc at D. Join OD and produce it to E.*



Then, **∠AOE = 120°**.

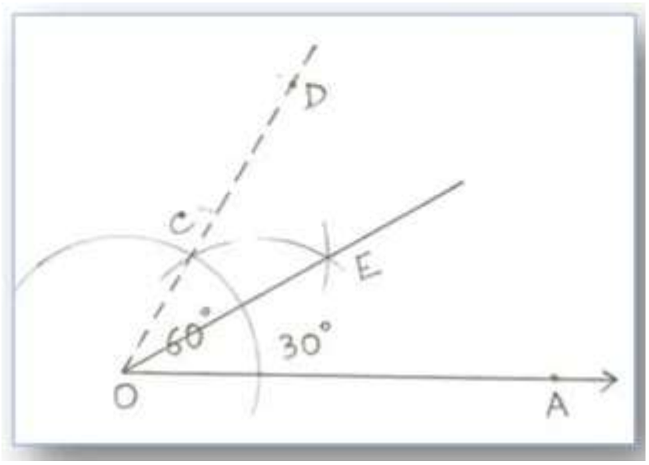
CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

3. Construction of an Angle of 30° by using Compass

Step of Construction:

(i) *Construction an angle ∠AOD = 60° as shown.*

(ii) *Draw the bisector OE of ∠AOD.*



Then, **∠AOD = 30°**.

4. Construction of an Angle of 90° by using Compass

Step of Construction:

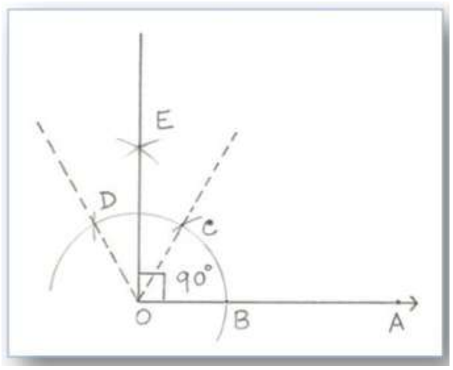
(i) *Take any ray OA.*

(ii) *With O as centre and any convenient radius, draw an arc cutting OA at B.*

(iii) *With B as centre and the same radius, draw an cutting the first arc at C.*

(iv) *With C as centre and the same radius, cut off an arc cutting again the first arc at D.*

(v) *With C and D as centre and radius of more than half of CD, draw two arcs cutting each other at E, join OE.*



Then, **∠EOA = 90°**.

5. Construction of an Angle of 75° by using Compass

Step of Construction:

(i) *Take a ray OA.*

(ii) *With O as centre and any convenient radius, draw an arc cutting OA at C.*

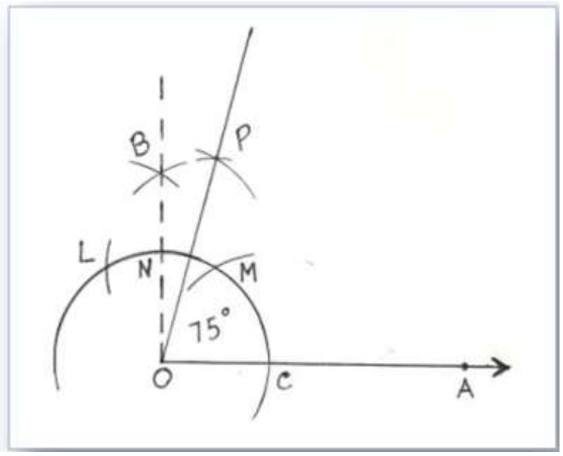
(iii) *With C as centre and the same radius, draw an cutting the first arc at M.*

(iv) *With M as centre and the same radius, cut off an arc cutting again the first arc at L.*

(v) *With L and M as centre and radius of more than half of LM, draw two arcs cutting each other at B, join OB which is making 90°.*

(vi) *Now with N and M as centres again draw two arcs cutting each other at P.*

(vii) *Join OP.*



Then, **∠POA = 75°**.

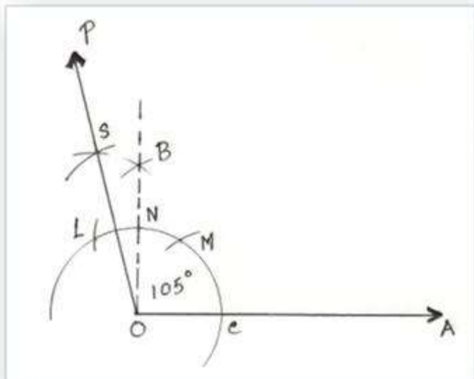
CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

6. Construction of an Angle of 105° by using Compass

Step of Construction:

(i) *After making 90° angle take L and N as centre and draw two arcs cutting each other at S.*

(ii) *Join SO.*



Then, **∠SOA = 105°**.

7. Construction of an Angle of 135° by using Compass

Step of Construction:

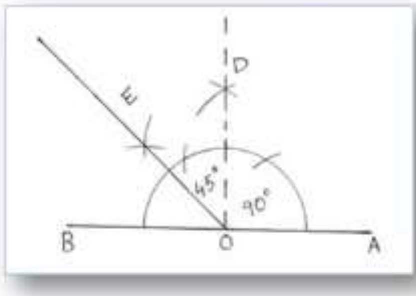
(i) *Construct ∠AOD = 90°*

(ii) *Produce ∠AO to B.*

(iii) *Draw OE to bisect ∠DOB.*

∠DOE = 45°

∠EOA = 45° + 90° = 135°



Then, **∠EOA = 135°**.

8. Construction of an Angle of 150° by using Compass

Step of Construction:

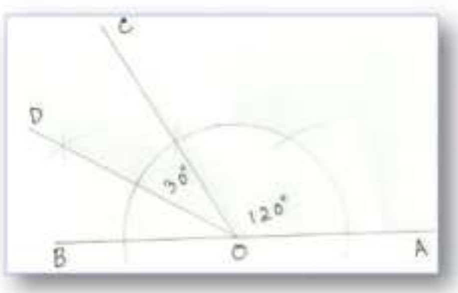
(i) *Construct ∠AOC = 120°*

(ii) *Produce ∠AO to B.*

(iii) *Draw OD to bisect ∠COB.*

Now ∠COD = 30°

Therefore, ∠AOD = 120° + 30° = 150°



Then, **∠AOD = 150°**.

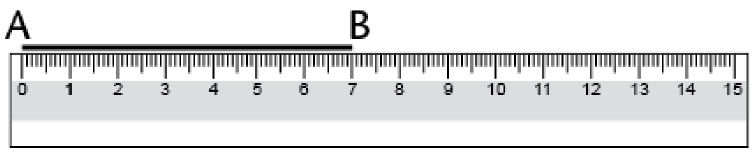
CONSTRUCTING TRIANGLES WITH GIVEN ANGLES AND LINE SEGMENTS;

Constructing triangles when three sides are given

1. Read through the following steps. They describe how to construct $\triangle ABC$ with side lengths of 3 cm, 5 cm and 7 cm.

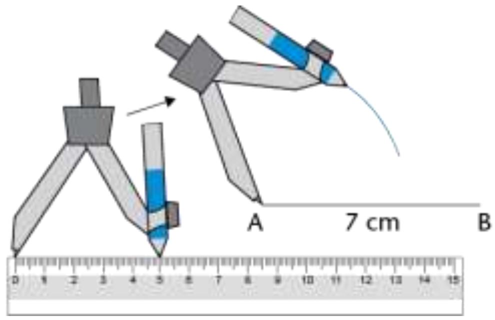
Step 1

Draw one side of the triangle using a ruler. It is often easier to start with the longest side.



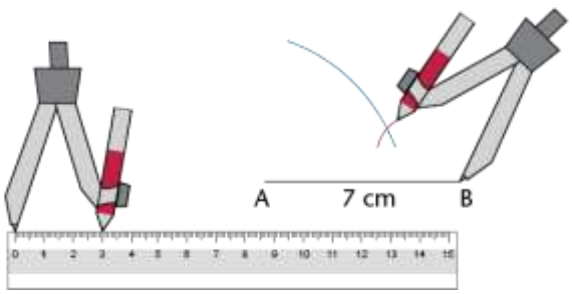
Step 2

Set the compass width to 5 cm. Draw an arc 5 cm away from point A. The third vertex of the triangle will be somewhere along this arc.



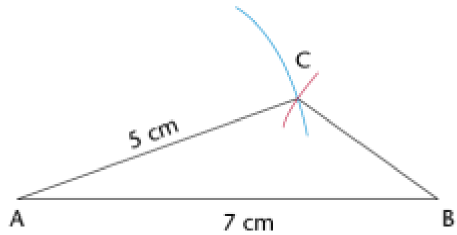
Step 3

Set the compass width to 3 cm. Draw an arc from point B. Note where this arc crosses the first arc. This will be the third vertex of the triangle.



Step 4

Use your ruler to join points A and B to the point where the arcs intersect (C).

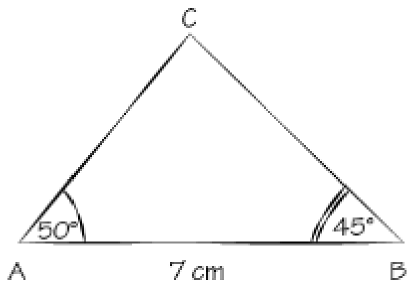


2. Work in your exercise book. Follow the steps above to construct the following triangles:
   1. \(\triangle ABC\) with sides 6 cm, 7 cm and 4 cm
   2. \(\triangle KLM\) with sides 10 cm, 5 cm and 8 cm
   3. \(\triangle PQR\) with sides 5 cm, 9 cm and 11 cm

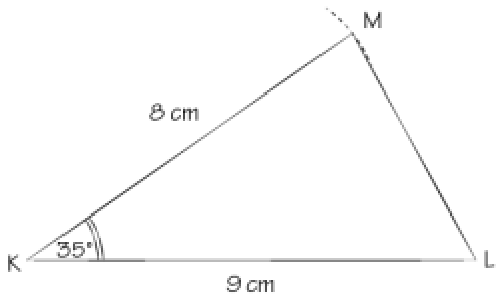
Constructing triangles when certain angles and sides are given

3. Use the rough sketches in (a) to (c) below to construct accurate triangles, using a ruler, compass and protractor. Do the construction next to each rough sketch.
   - The dotted lines show where you have to use a compass to measure the length of a side.
   - Use a protractor to measure the size of the given angles.

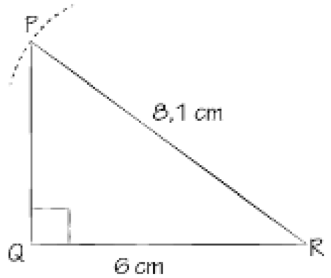
3. Construct \(\triangle ABC\), with **two angles and one side given**.



4. Construct a \(\triangle KLM\), with **two sides andan angle given**.



5. Construct right-angled \(\triangle PQR\), with the**hypotenuse and one other side given**.



Measure the missing angles and sides of each triangle in 3(a) to (c) on the previous page. Write the measurements at your completed constructions.

Compare each of your constructed triangles in 3(a) to (c) with a classmate's triangles. Are the triangles exactly the same?

CONSTRUCTING POLYGONS

REGULAR POLYGONS

A **regular polygon** is a polygon that is **equiangular** and **equilateral**. This means that all its angles are the same measure and all its sides are the same length.

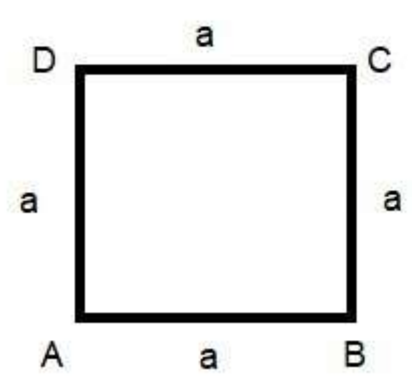
Constructing regular polygons

We know that **a regular polygon** is a polygon that has all sides of equal length and all interior angles of equal measure. In this lesson we'll learn how to construct them using compass and a ruler.

Square

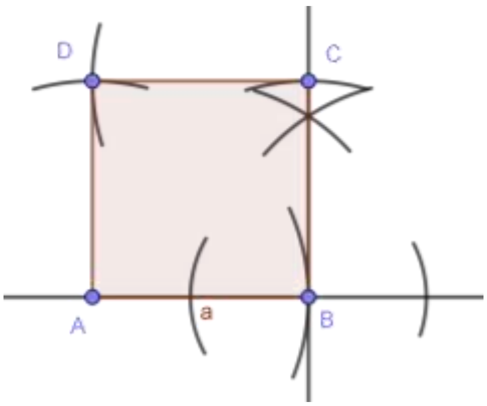
Example. Construct a square if we know the side a.

First, we make a sketch to know the disposition of the points and sides.



We will start by creating a line and a point A. Then we take the length of the side a in the width of the compass and make an arc that intersects with the line we drew first. The intersection is point B, our second vertex. Now we need to construct a line that's perpendicular to the AB, with the B being the point of intersection. We do this because we want to create the angle ∡ABC=90∘.

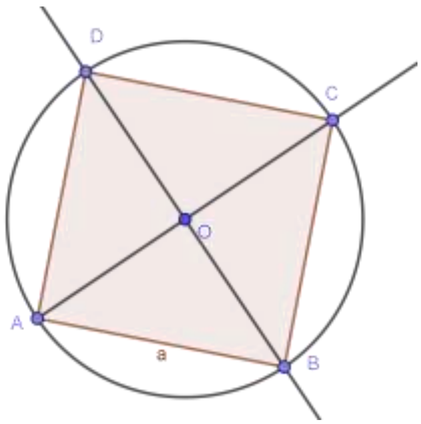
Now we have a perpenticular line, and we know the vertex C will be on it. Take the length of the side a in the width of the compass and make an arc that intersects with the perpendicular line – the intersection is our vertex C. What's left is constructing the vertex D. We do that by creating two arcs of the circles c(A,a) and c(C,a). Their intersection is the final vertex, vertex D.



* We only constructed one square here, but we could have constructed four of them by following the same process we did in construction of equilateral triangle.

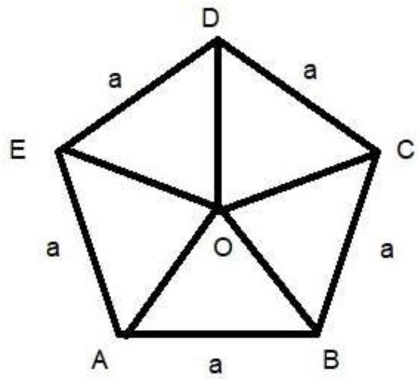
Example. How to construct a square if we know the radius of it's circumcircle?

First we draw a point O and a circle c(O,r). Pick a starting point A anywhere on the circle. Now let's draw a diameter from A through O. Let the point of intersection of a diameter and a circle be the point C. The line AB is one diagonal of the square we want to construct. How to get the other one? We know that diagonals in square are perpendicular, so we create a perpendicular line to the diameter AC. Make sure that the point of intersection is the point O. Now, the points of intersection of perpendicular and a circle c(O,r) are points B and C, our two last vertices.



Regular pentagon

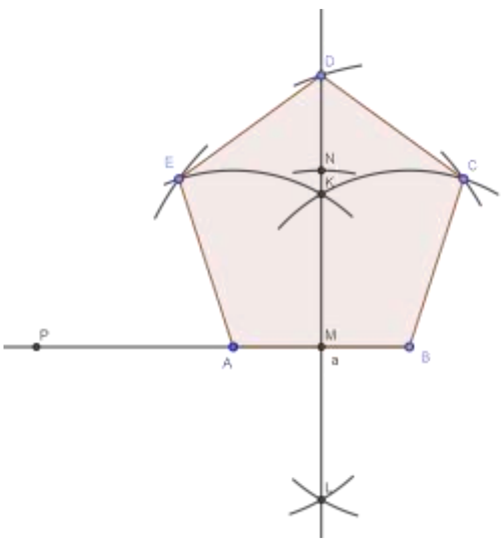
First we draw a sketch by hand. It doesn't have to be perfect since it's not our final construction, we'll just use it for planing.



Example. Construct a regular pentagon if we know the side a.

Make a ray with B being its endpoint and then construct point A so that |AB|=a. We want to create a bisector of |AB|. Take the compass and make sure the width of compass is the length of side a (IMPORTANT!). Put the needle on B and make two arcs of the circle c(B,a). Repeat the step for arcs of c(A,a). The arcs intersect in points K and L. Join them to get the midpoint between A and B, point M. Again, keep the compass radius the length of a, put the compass' needle on M and make an arc that intersects with bisector line, making the point N. Now, adjust the compass to the length of AN. Put the needle in A and make an arc that intersects with the ray we made at the beginning, that will give us a point P.

The distance from M to P is very important distance – it will give us the rest of the vertices. Make the compass' radius equal to the distance between M and P. Put the needle on B and make an arc that intersects with one of the arcs we made to get midpoint. Make the second arc that intersects with the bisector line. The intersects will be points E and D respectively. To get the vertex C we will put the needle in A and repeat the process.



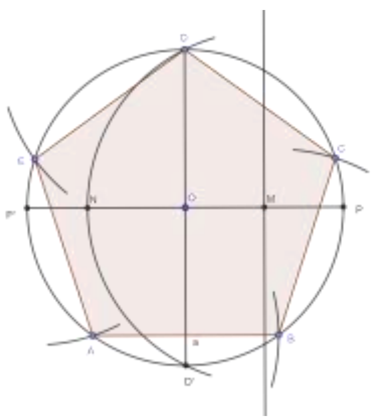
CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Example. Construct a regular pentagon if we know the radius of the circumscribed circle.

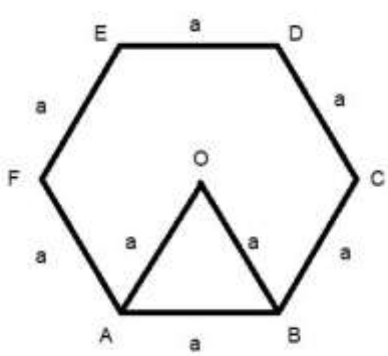
Construct the circle c(O,r) and two perpendicular diameters, AA′— and PP′— . Now lets construct the bisector of segment OP—, the intersection will be point M. In the width of compass take the length between A and M, and put a compass needle on M to create an arc that intersects with PP′—. The intersection is N.

The distance from A to N is the length of the side a of regular pentagon. Now that we know the length of a, we need to construct vertices. Let D be our first vertex.

Firstly, we open the compass to the length of a and put the needle of compass on A. Now make an arc that intersects with the circle c(O,r|) giving us the vertex B. Without changing the width of compass, we put the needle on B and do the same process to get vertex A, and so on. This process will give us the remaining 4 vertices.



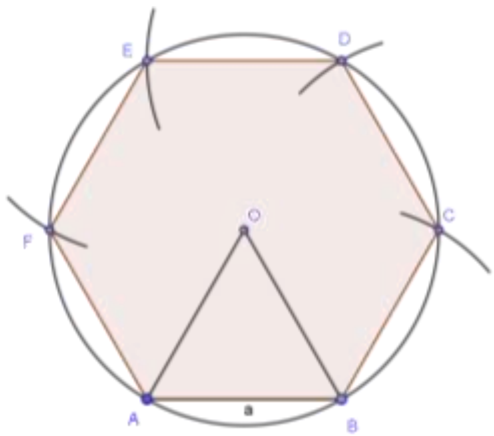
Regular hexagon



Example. Construct a regular hexagon if we know side a.

We can break regular hexagon into 6 equilateral triangles with the side a. The vertex O is the center of inscribed and circumscribed circles, and |AO|=|BO|=|CO|=|DO|=|EO|=|FO|. First we construct △ABO following the proces we used in constructing equilateral triangle. Let's draw c(O,|AO|). Since O is the center of circumscribed circle, we know that hexagon's vertices will be on the circle. Now we just take the length of a into the width of a compass and make 4 arcs on the circle.

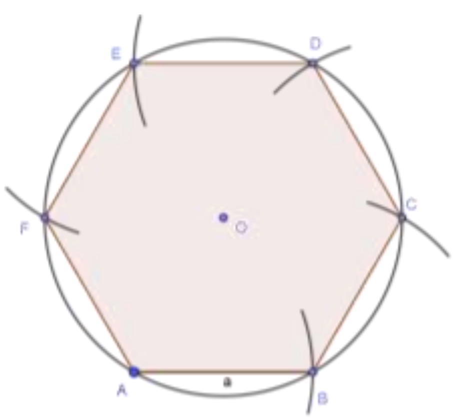
Without changing the width of compass we put the needle of compass on the B, make and arc that intersects with the circle c(O,|AO|) giving us the vertex C. Then we put the needle on C and do the same process to get vertex D, and so on. This process will give us the last 4 vertices. It doesn't matter if we start from A, and do it clockwise or from B like we did here, the result will be the same.



Important to remember: In regular hexagon △ABO is equilateral triangle, which means that the length of a radius of circumscribed circle and the length of a side are always equal.

Example. Construct a regular hexagon if we know the radius of circumscribed circle.

This is even easier. We just draw the circle c(O,r) and pick a starting point on it, let it be point A. In regular hexagon, we know that the radius of circumscribed circle is equal to side of polygon, meaning r=|AO|=|AB|. Now that we have the side of a regular hexagon, we construct the remaining 5 vertices like we did in the last example.

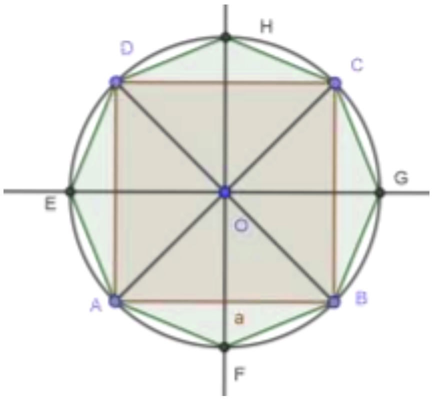


CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Regular octagon

Example. How to construct a regular octagon if we know radius of circumscribed circle?

First we construct a square within the given circle with its diagonals following the process described above. Then we construct angle bisectors of ∡AOB,∡BOC,∡COD and ∡DOA. Bisectors instersect with the circumscribed circles giving us 4 new points, E,F,G and H. Those points are the remaining vertices of an octagon.

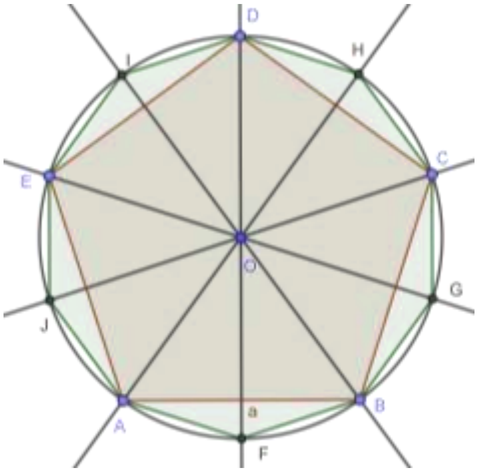


Regular decagon

Example. How to construct a regular decagon if we know radius of circumscribed circle?

First we construct a regular pentagon within the given circle following the process described in construction of pentagon. Then we connect each vertex to the center of circumscribed circle to divide pentagon into 5 congruent triangles. The next step is constructing construct angle bisectors of ∡AOB,∡BOC,∡COD,∡DOE and ∡EOA.

Bisectors intersect with the circumscribed circles giving us 5 new points, F,G,H,I and J. Those points are the remaining vertices of an decagon.

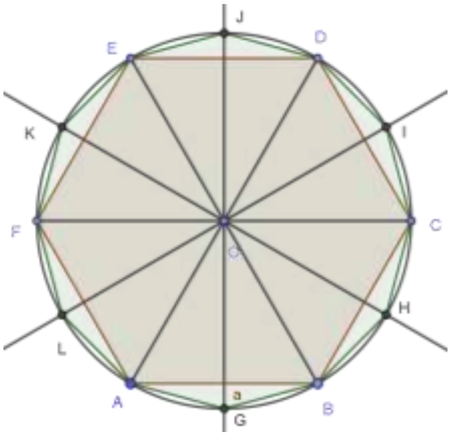


Regular dodecagon

Example. How to construct a regular dodecagon if we know radius of circumscribed circle?

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Just like we did in the last two examples, we can construct regular dodecagon out of a regular hexagon. Angle bisectors of central angles of a hexagon give us remaining vertices of dodecagon.



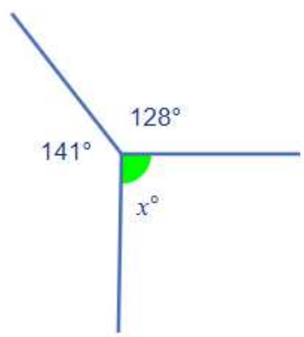
UNIT 3
ANGLES AND POLYGONS

ANGLES AT A POINT

Angles around a point add up to 360°. This fact can be used to calculate missing angles.

HOW TO FIND MISSING ANGLES AROUND A POINT

Calculate the value of x.



I know that there are 360° in a full turn.

I can see that there is already an angle of 128° and 141°.

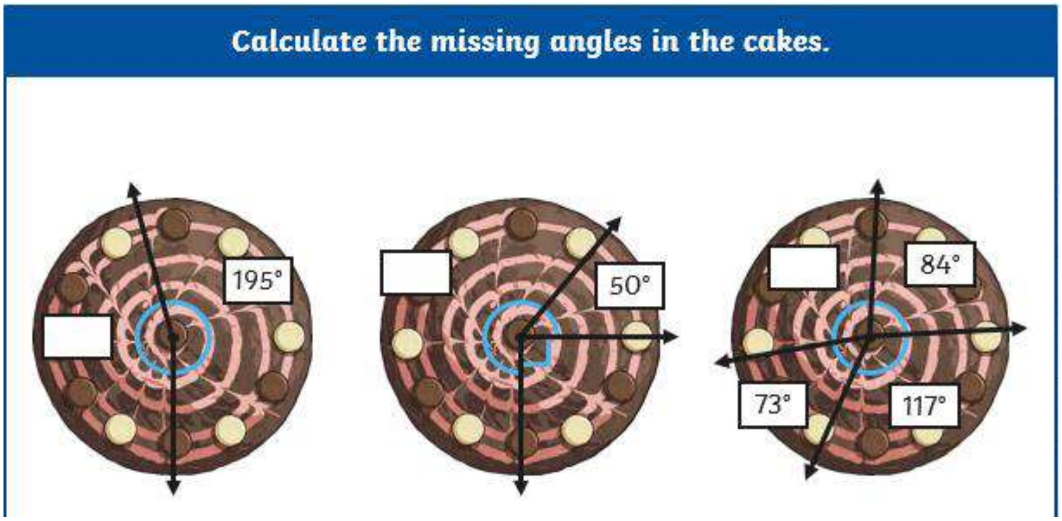
128 + 141 = 269

To find the missing angle I need to find the difference between 269 and 360.

360 – 269 = 91

Angle x = 91°

WHY DON'T YOU TRY?

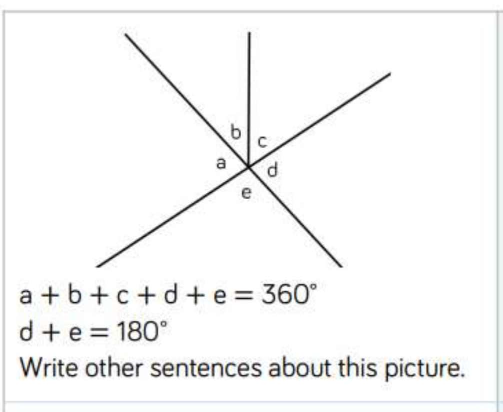


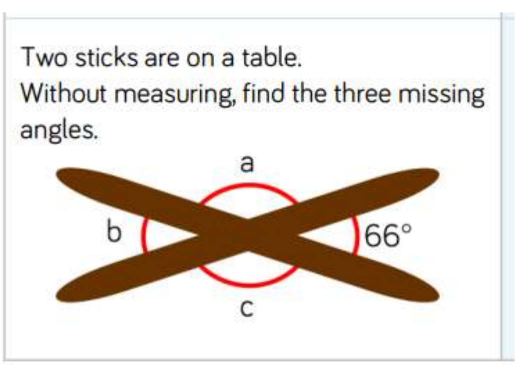
Remember that all of the angles should add up to 360°.

Add together the angles you already have.

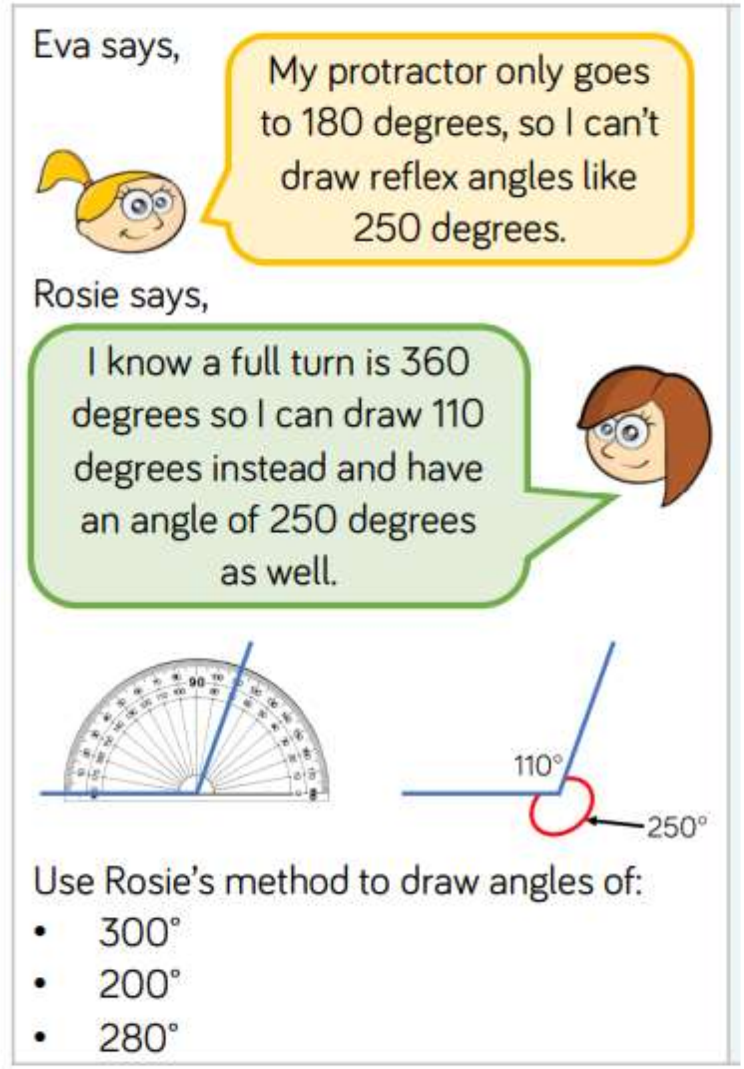
Then find the difference between 360.

WHY NOT HAVE A GO AT SOME OF THESE REASONING AND PROBLEM SOLVING QUESTIONS





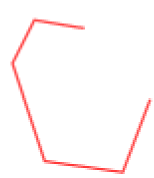
CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)



SUM OF INTERIOR ANGLES IN A POLYGON

IS IT A POLYGON?

Polygons are 2-dimensional shapes. They are made of straight lines, and the shape is "closed" (all the lines connect up).



Polygon (straight sides)

Not a Polygon (has a curve)

Not a Polygon (open, not closed)

Polygon comes from Greek. **Poly-** means "many" and **-gon** means "angle".

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Types of Polygons

Regular or Irregular

A **regular** polygon has all angles equal and all sides equal, otherwise it is **irregular**

Regular Irregular

Concave or Convex

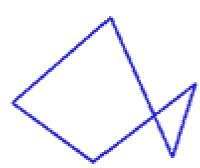
A **convex** polygon has no angles pointing inwards. More precisely, no internal angle can be more than 180°.

If any internal angle is greater than 180° then the polygon is **concave**. (*Think: concave has a "cave" in it*)

Convex Concave

Simple or Complex

A **simple** polygon has only one boundary, and it doesn't cross over itself. A **complex** polygon intersects itself! Many rules about polygons don't work when it is complex.

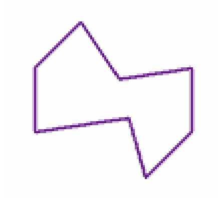


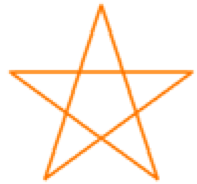
Simple Polygon
(this one's a Pentagon)

Complex Polygon
(also a Pentagon)

More Examples

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

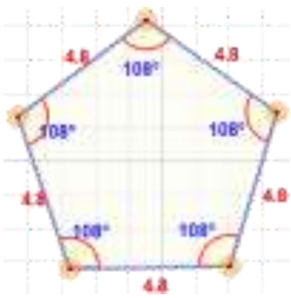




Irregular Hexagon

Concave Octagon

Complex Polygon
(a "star polygon",
in this case a pentagram)



Names of Polygons

			If it is a Regular Polygon...
Name	**Sides**	**Shape**	**Interior Angle**
Triangle *(or Trigon)*	**3**		60°
Quadrilateral *(or Tetragon)*	**4**		90°
Pentagon	**5**		108°
Hexagon	**6**		120°
Heptagon *(or Septagon)*	**7**		128.571°
Octagon	**8**		135°

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Nonagon *(or Enneagon)*	**9**		140°
Decagon	**10**		144°
Hendecagon *(or Undecagon)*	**11**		147.273°
Dodecagon	**12**		150°
Triskaidecagon	**13**		152.308°
Tetrakaidecagon	**14**		154.286°
Pentadecagon	**15**		156°
Hexakaidecagon	**16**		157.5°
Heptadecagon	**17**		158.824°
Octakaidecagon	**18**		160°
Enneadecagon	**19**		161.053°
Icosagon	**20**		162°
Triacontagon	**30**		168°
Tetracontagon	**40**		171°
Pentacontagon	**50**		172.8°
Hexacontagon	**60**		174°
Heptacontagon	**70**		174.857°
Octacontagon	**80**		175.5°
Enneacontagon	**90**		176°
Hectagon	**100**		176.4°
Chiliagon	**1,000**		179.64°
Myriagon	**10,000**		179.964°
Megagon	**1,000,000**		~180°

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Googolgon	**10^{100}**		~180°
n-gon	**n**		(**n**−2) × 180° / **n**

INTERIOR ANGLE FORMULA (DEFINITION, EXAMPLES, SUM OF INTERIOR ANGLES)

If you take a look at other geometry lessons on this helpful site, you will see that we have been careful to mention interior angles, not just angles, when discussing polygons. Every polygon has interior angles and exterior angles, but the interior angles are where all the interesting action is.

INTERIOR ANGLE FORMULA

From the simplest polygon, a triangle, to the infinitely complex polygon with nn sides, sides of polygons close in a space. Every intersection of sides creates a vertex, and that vertex has an interior and exterior angle. **Interior angles of polygons** are within the polygon.

Though Euclid did offer an exterior angles theorem specific to triangles, no Interior Angle Theorem exists. Instead, you can use a formula that mathematically describes an interesting pattern about polygons and their interior angles.

SUM OF INTERIOR ANGLES FORMULA

This formula allows you to mathematically divide any polygon into its minimum number of triangles. Since every triangle has interior angles measuring 180°180°, multiplying the number of dividing triangles times 180°180° gives you the sum of the interior angles.

S = (n − 2) × 180°S = (n - 2) × 180°

S = sum of interior anglesS = sum of interior angles

n = number of sides of the polygonn = number of sides of the polygon

Try the formula on a triangle:

S = (n − 2) × 180°S = (n - 2) × 180°

S = (3 − 2) × 180°S = (3 - 2) × 180°

S = 1 × 180°S = 1 × 180°

S = 180°S = 180°

Well, that worked, but what about a more complicated shape, like a dodecagon?

[insert dodecagon drawing]

It has 12 sides, so:

S = (n − 2) × 180°S = (n - 2) × 180°

S = (12 − 2) × 180°S = (12 - 2) × 180°

S = 10 × 180°S = 10 × 180°

S = 1,800°S = 1,800°

How do you know that is correct? Take any dodecagon and pick one vertex. Connect every other vertex to that one with a straightedge, dividing the space into 10 triangles. Ten triangles, each 180°180°, makes a total of 1,800°1,800°!

FINDING AN UNKNOWN INTERIOR ANGLE

The same formula, S = (n − 2) × 180°S = (n - 2) × 180°, can help you find a missing interior angle of a polygon. Here is a wacky pentagon, with no two sides equal:

[insert drawing of pentagon with four interior angles labeled and measuring 105°, 115°, 109°, 111°; length of sides immaterial]

The formula tells us that a pentagon, no matter its shape, must have interior angles adding to 540°540°:

S = (n − 2) × 180°S = (n - 2) × 180°

S = (5 − 2) × 180°S = (5 - 2) × 180°

S = 3 × 180°S = 3 × 180°

S = 540°S = 540°

So subtracting the four known angles from 540°540° will leave you with the missing angle:

540° − 105° − 115° − 109° − 111° = 100°540° - 105° - 115° - 109° - 111° = 100°

The unknown angle is 100°100°.

FINDING THE NUMBER OF SIDES OF A POLYGON

You can use the same formula, S = (n − 2) × 180°S = (n - 2) × 180°, to find out how many sides nn a polygon has, if you know the value of SS, the sum of interior angles.

You know the sum of interior angles is 900°900°, but you have no idea what the shape is. Use what you know in the formula to find what you do not know:

State the formula:

S = (n − 2) × 180°S = (n - 2) × 180°

Use what you know, S = 900°S = 900°

900° = (n − 2) × 180°900° = (n - 2) × 180°

Divide both sides by 180°180°

900°180° = ((n − 2) × 180°)180°900°180° = ((n - 2) × 180°)180°

No need for parentheses now

5 = n − 25 = n - 2

Add 2 to both sides

5 + 2 = n − 2 + 25 + 2 = n - 2 + 2

7 = n7 = n

The unknown shape was a heptagon!

Lesson Summary

Now you are able to identify interior angles of polygons, and you can recall and apply the formula, S = (n − 2) × 180°S = (n - 2) × 180°, to find the sum of the interior angles of a polygon. You also are able to recall a method for finding an unknown interior angle of a polygon, by subtracting the known interior angles from the calculated sum.

Not only all that, but you can also calculate interior angles of polygons using SnSn, and you can discover the number of sides of a polygon if you know the sum of their interior angles. That is a whole lot of knowledge built up from one formula, S = (n − 2) × 180°S = (n - 2) × 180°

FINDING THE VALUE OF AN INTERIOR ANGLE OF A GIVEN REGULAR POLYGON

FINDING INTERIOR ANGLES OF REGULAR POLYGONS

Once you know how to find the sum of interior angles of a polygon, finding one interior angle for any regular polygon is just a matter of dividing.

Where SS = the sum of the interior angles and nn = the number of congruent sides of a regular polygon, the formula is:

SnSn

Here is an octagon (eight sides, eight interior angles). **First, use the formula for finding the sum of interior angles:**

S = (n − 2) × 180°S = (n - 2) × 180°

S = (8 − 2) × 180°S = (8 - 2) × 180°

S = 6 × 180°S = 6 × 180°

S = 1,080°S = 1,080°

Next, divide that sum by the number of sides:

- measure of each interior angle = Sn= Sn
- measure of each interior angle = 1,080°8= 1,080°8

- measure of each interior angle = = 135°= 135°

Each interior angle of a regular octagon is = 135°= 135°.

RELATIONSHIP BETWEEN EXTERIOR AND INTERIOR ANGLES OF A POLYGON.

Diagonals

One property of all convex polygons has to do with the number of diagonals that it has:

Every convex polygon with n sides has n(n-3)/2 diagonals.

With this formula, if you are given either the number of diagonals or the number of sides, you can figure out the unknown quantity. Diagonals become useful in geometric proofs when you may need to draw in extra lines or segments, such as diagonals.

Figure %: Diagonals of polygonsThe figure with 4 sides, above, has 2 diagonals, which accords to the formula, since 4(4-3)/2 = 2. The figure with 8 sides has twenty diagonals, since 8(8- 3)/2 = 20.

Interior Angles

The interior angles of polygons follow certain patterns based on the number of sides, too. First of all, a polygon with n sides has n vertices, and therefore has n interior angles. The sum of these interior angles is equal to 180(n-2) degrees. Knowing this, given all the interior angle measures but one, you can always figure out the measure of the unknown angle.

Exterior Angles

An exterior angle on a polygon is formed by extending one of the sides of the polygon outside of the polygon, thus creating an angle supplementary to the interior angle at that vertex. Because of the congruence of vertical angles, it doesn't matter which side is extended; the exterior angle will be the same.

The sum of the exterior angles of any polygon (remember only convex polygons are being discussed here) is 360 degrees. This is a result of the interior angles summing to 180(n-2) degrees and each exterior angle being, by definition, supplementary to its interior angle. Take, for example, a triangle with three vertices of 50 degrees, 70 degrees, and 60 degrees. The interior angles sum to 180 degrees, which equals 180(3-2). Because the exterior angles are supplementary to the interior angles, they measure, 130, 110, and 120 degrees, respectively. Summed, the exterior angles equal 360 degreEs.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

A special rule exists for regular polygons: because they are equiangular, the exterior angles are also congruent, so the measure of any given exterior angle is 360/n degrees. As a result, the interior angles of a regular polygon are all equal to 180 degrees minus the measure of the exterior angle(s).

Notice that the definition of an exterior angle of a polygon differs from that of an exterior angle in a plane. A polygon's exterior angle is not equal to 360 degrees minus the measure of the interior angle. A polygon's interior and exterior angles at a given vertex don't span the entire plane, they only span half the plane. That is why they are supplementary--because their measures sum to 180 degrees instead of 36

UNIT 4

DIAGNOSIS AND REMEDIATION; ASSESSMENT RESOURCES/RECORDS, AND MONITORING PROGRESS

MISCONCEPTION DIAGNOSIS

CLASSROOM ASSESSMENT RESOURCES AND RECORDS

INTERPRETING DATA/REPORTS ON PERFORMANCE AND PROVIDING FEEDBACK

EVALUATING PERFORMANCE AND MONITORING PROGRESS

UNIT 5
MICRO LESSONS AND USE OF TECHNOLOGY ACROSS JUNIOR HIGH SCHOOL MATHEMATICS

WHAT IS A MICRO LESSON?

Micro lessons are bite-sized modules that focus only on key elements or messages of a learning topic. Unlike traditional modules that take hours to get completed, micro lessons are designed for self-paced learning that can be completed only within five to ten minutes. This way, training managers can prevent cognitive overload in which the brain is forced to digest abundant information all at once, negatively affecting knowledge retention among learners. Through micro lessons, information can be effectively embedded into the long-term memory of learners while also empowering them to have control over their learning process. This makes training not only more understandable but also more engaging and less time-consuming.

WHAT IS MICRO TEACHING?

Micro teaching is a proven method so you can micro teach to the best of your abilities and ultimately drive better learning results. Simply put, micro teaching involves scaling back the lesson material so that any given team member can absorb what's being taught in small bursts. This method is a popular one not only because of the brevity of lessons, but because microlearning has also proven to help convert short-term memory to long term, meaning that delivering learning material in bite-sized chunks can be incredibly effective, if not the *most* effective form of learning. If you're unfamiliar with microlearning, it's the breakdown of information into topical, bite-sized chunks, not only making it easier for the learner to absorb the content but also resulting in better learning. In fact, microlearning sees completion rates around the 90% mark (where completion rates are commonly around 15% in traditional eLearning). What's more, these courses are interactive and fun, which means learners actually enjoy completing the learning content. According to a recent study, 94% of L&D professionals say that they prefer microlearning to traditional eLearning courses because their learners prefer it (Boyette).

Apart from being modern and fun, one of the reasons why learners prefer microlearning is because it allows them to easily digest the information rather than being overwhelmed with too much to take in. By the same token, micro teaching delivers micro lesson(s) in small bursts to maximize absorption and mitigate any unnecessary content to provide the best learning results possible.

WHAT IS A MICRO LESSON PLAN?

Simply put, a micro lesson plan is focused on one specific subject to be explored within a learning platform. Your micro lesson plan should be designed with the intention of carrying out short, succinct lessons for your learners to master.

IMPORTANCE OF LESSON PLANNING

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Lesson-Planning has certain definite functions to perform which are indispensable in good teaching.

1. Lesson- Planning gives the teacher greater assurance and greater freedom in teaching. The teacher who has planned his lesson wisely, enters the class-room without anxiety, ready to embark with confidence upon a job he understands and prepared to carry it to a workman like conclusion.

2. It provides for adequate lesson summaries, ensures a definite assignment for class, and availability of materials for lesson when needed.

3. It stimulates the teacher to introduce pivotal questions and illustrations.

4. Since lesson planning establishes proper connections between different lessons or units of study, it provides and encourages continuity in the teaching provides and encourages continuity in the teaching process.

5. It ensures association between various lessons in the same main, unit, the selection and organisation of subject-matter, materials and activities.

6. It enables the teacher to know the most desirable type of teaching procedures and to prepare tests of progress and checks for judging the outcomes of instruction.

7. Lesson-planning prevents waste because it helps the teacher to be systematic and orderly. It saves him form haphazard teaching.

MICRO LESSON PLANNING FORMATS

MICROTEACHING LESSON PLAN Time: 15 min. **Name:** Crystal Tse **Student ID #:** XXXXXXXX **Attended Effective Lesson Planning workshop:** Yes **Lesson Title:** Stereotype Threat: Why we need identity-safe environments **Please check the topic of the lesson** Your favourite theory or theorist in your field of study. **Learning objectives (list 1-2 specific objectives):** *(What will learners know or be able to do after your lesson?)* ***Hint:*** *Consult Bloom's taxonomy to select appropriate action verbs.* 1. Explain what is meant by stereotype threat, and how this affects people's performance. 2. Propose an intervention to help alleviate the effects of stereotype threat.
Pre-assessment: *(If you plan to find out what learners already know about your topic, how will you do it?)*

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

I will be assuming that most students will not know what stereotype threat is, and so the video clip and brainstorming activity (see below) will be used to introduce the topic to students.
Opening your lesson: *(How will you get learners' attention and motivate them?)*   I will show them a 1 minute video clip from the television show, The Office, where they parody diversity training. Different people in the office have to take on an ethnic identity and treat others based on that identity. I want the class to think about different stigmatized identities, and how people might feel when they become aware of the negative stereotypes associated with the group to which they belong. Duration: 1 minute
Learner engagement and participation: *(How will you engage your audience with your topic and encourage their participation in the lesson?)* Describe specific learning activities and interaction you are planning.   I will ask the class to brainstorm different reasons why a minority student is not performing well in her class (I will give a more detailed example explaining the student's situation).   Then I will ask them to imagine what it feels like to experience stereotype threat (I will guide them through a short thought experiment), and propose that the experience of stereotype threat is one reason why minority or negatively stereotyped students

FORMAT OF LESSON PLAN

Micro-Lesson Plan-1

Skill of Engagement

Name of Trainee:.................................Roll No................

Class...............Subject.........................Duration..............

Topic...

Module	Activity plan	Component Used
States of Matter	Teacher will ask the students about matter by using following questions; 1. What do you mean by the term matter? 2. Give some examples of matter from your surroundings.	Integration with previous knowledge
	Teacher will show some real objects/things present in the class, such as; Duster, chair, water, etc. and ask them to recognize whether it is a solid, liquid or gas.	Interaction with students Gaining attention
	Teacher will make the students clear that these are the different states of matter as;	Maintaining continuity in content
	Teacher will show the molecular arrangement of different states of matter using a chart.	Use of devices and techniques/multimedia
	Teacher will ask the students represent three states of matter by drawing a flowchart on their notebooks.	Gaining attention

DESIGN OF MICRO LESSONS

The goal of teaching refers to the focus of this micro-lessons content and to achieve the results. As the micro-lessons time to control in 20 min, try to split into a number of sub-courses, to ensure that short and pithy. To "vitamin A and human health", for example, the micro-lessons mainly teach the source of vitamin A, biochemical and lack of disease. We take into account the extent of the audience's understanding and close contact with daily life and the symptoms of common diseases such as night blindness or dry eye.

MICRO TEACHING IDEAS

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

If you're unsure where to start, we've collected 10 ideas for micro teaching to get you started:

1. **Onboarding**. If you're onboarding new staff, one of the best ways to do this is through micro teaching. Simply scale back your onboarding process by breaking it up into small, digestible chunks. In an EdApp microlesson, you can easily introduce your company and your teams, values and culture, and more. It also gives new starters the opportunity to revisit their onboarding lesson if they missed anything
2. **Compliance.** Have any company-wide or even team-wide compliance-based training? Scale back your content into bite-sized portions so it can easily fit into a microlesson. For reference, it should take around 5 – 10 minutes to complete one microlesson.
3. **Product training.** Especially if you're in an industry like retail, create a micro teaching plan around your products or collections. Microlearning allows you to easily break down your products into categories and simultaneously boosts retention rates.
4. **Health and Safety.** Whether you're in an office, on-site, or working remotely, every organization has best practices around health and safety. Easily upload microlessons around your fire escape plan, educate your teams on COVID-19, and everything in between.
5. **Team Building.** Build and maintain your company culture through team-building exercises. Include anything from breakout activities, games, and engagement to build happy, well-functioning teams in your workplace.
6. **Sustainability.** Looking to do your part by educating your teams on sustainability practices they can introduce to their lives at home and at work? Why not share sustainability content on everything from the United Nations' Sustainable Development Goals (SDGs) to sustainable eating and more.
7. **Leadership skills.** Empower your teams with the skills they need to be great leaders in the workplace. Like team building micro teaching, leadership skills will help nurture and grow your teams by giving them the tools they need to thrive.
8. **Communication skills.** Regardless of your industry, communication skills are imperative to help drive the success of your business. Train your teams on verbal, written, and non-verbal communication tactics and styles to nurture professional growth.
9. **IT & Security.** Keeping your teams safe and secure is key. Educate on cybersecurity awareness and privacy to help protect you and your teams and share IT procedures to support your teams.
10. **Mental health.** Health and wellbeing are key for every team. Share micro teaching on mental health and resilience in EdApp's editable content library, provided by industry leaders around the globe. Pick and choose which content you would like to share with your teams to facilitate health and balance.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

WHAT DIFFERS BETWEEN A MICRO LESSON AND A MACRO LESSON PLAN?

So what is the difference between a micro lesson plan and a macro lesson plan? Simply put, a micro lesson plan focuses on brief subjects and incorporates a specific topic for learners to absorb quickly, without taking up too much time.

We know that human memory can only hold up to five new pieces of information before it gets lost or overwritten. This is one of the reasons why microlearning is so successful by focusing on fewer topics and embedding the learned topics into long-term memory.

Alternatively, a macro lesson plan is a teaching strategy that incorporates a magnitude of lessons and subjects, designed to be executed across a longer period of time. As the name suggests, a macro lesson plan is also more suited to teach larger concepts which often consist of more complex and detailed subject matter.

How to make the best micro lesson plan in 4 steps

So, what's the best way to approach an amazing micro lesson plan? Our learning experts recommend breaking your plan down into clear steps to ensure that you're creating the best learning experience for your teams. Read on to discover our 4-step guide and learn how to expand your micro teaching ideas.

1. Introduce learners to the topic with a title slide

It's important to start by telling your learners what the lesson is about. This puts them in the right frame of mind by getting them thinking about learning and making them more receptive to new information. By giving them an overview of the topic, you're also making them think about the context (for more on why this is important, see this article on chunking strategy) and what they know about the subject already. The more they can relate to the subject matter, the more effective the lesson will be, all while providing a greater opportunity for the new information to embed into your learners' long-term memory with ease. Pro tip: A simple introductory slide will do the job, but avoid presenting a wall of text as it will turn people off.

2. Begin knowledge transfer with video, text or both

Video is proven to be one of the best forms of knowledge transfer. Text is fine but, again, we recommend keeping it minimal to avoid turning off your learner. Using five-or-six content slides in a row, we find, is too many for a micro lesson plan. If all of the information is important, consider splitting the information into multiple lessons. We find using four slides is an optimal number for directed-focus lessons.

You can easily find perfect examples of a balance of video, text, and both in our Editable Content Library, where leading companies around the world have contributed high-quality, practical courseware – all for free.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

For example, the United Nations Institute for Training and Research (UNITAR) has created a collection of brilliant lessons, including *Meet the Sustainable Development Goals*. Accessible now for all to take, this course introduces the 2030 Agenda for Sustainable Development and optimal understanding of the Sustainable Development Goals (SDGs). This learning content guides learners through a multi-part course to share amongst colleagues, friends, and family, for effective mainstreaming of the SDGs.

3. Micro teach to reinforce content using interactive questions and games

Next, our learning experts recommend introducing interactive questions to help reinforce your content. If your learners get the answer right, you can reinforce why the right answer was important. However, if they get it wrong, it's very important to quickly correct any misconception and explain what the right answer is: any delay will increase the likelihood of retaining the wrong information. Just as you would for answering the question correctly, it's important to tell your learners what the correct answer is and why the information is important – this will leave them with a lingering takeaway message. EdApp boasts an expansive library of over 50 templates to help you create interactive questions or games to help engage your learners and reinforce your concepts.

4. Applying gamification to a micro lesson plan

Playing games makes for effective learning, but making your lessons competitive (and even rewarding) will drive effectiveness even further. There are various microteaching methods for doing this (which ultimately depends on which Learning Management System you use) but scoring answers, setting time limits, and awarding points for completing tasks within the lesson all increase learners' engagement.

EdApp's built-in rewards program is built around earning 'stars'. Based on your preferences, you can easily reward your learners through stars, which can then be turned into real rewards like a Starbucks or Amazon gift card, for example.

Gamification and real prizing entice learners to collect more stars, as they complete your microlessons. Be generous though – don't give out one star after they've sat through 20 slides! Offering real prizes for best performance or simply completing a course on time naturally acts as a learning incentive.

A real-world example can be found in the realm of retail trainees: asking which statements about a product represent correct or incorrect things to say to customers – by swiping left for incorrect answers and swiping right for correct answers – gamifies the interactive learning, thereby improving retention.

Microlearning with built-in gamification training software is an amazing tool to adopt for the success of your organization's training strategy. We know that microlearning has proven successes and one of the best things about it, is that you have the freedom to implement the practice in many different ways, giving you the space to deploy custom, bespoke learning experiences to your audiences.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

CREATING THE INSTRUCTIONAL DESIGN

Understand the learner

Be sure to understand whether there are various types or levels of learners, their preferred modalities for learning, and the learner environment that could be encountered in the delivery of the microlearning lesson. On the Microlearning Course Outline, this can be addressed in sections "Learner Description", "Environment for Learning", "Desired Arousal States".

Know the big picture

It is helpful to know all learning gaps and to have a big picture of the subject landscape or processes which will be conveyed, in parts, through the microlearning lesson. Work with a mind-map to visualize the big picture – the content/learning landscape. On the Microlearning Course Outline form, this can be addressed in sections "Course Title", "What outcomes are candidates for consideration in the course/lesson?", "How does a proficient individual achieve the outcome now?", and "What are the current gaps to be addressed in the course/lesson?".

Begin with the end in mind

Microlearning focuses your lesson design on the achievement of "just one thing". Consider how to chunk the content so that you winnow away the "nice to haves" and get to the core outcome that must be achieved. On the Microlearning Course Outline form, this can be refined in sections entitled "Focused Outcome". Ensure this aligns with your inputs on the current means by which a proficient individual achieves the outcome.

Focus on the feeling

To be effective in any learning program, and microlearning is not exempt, you must consider what emotional states will get the learner engaged and support successful action on the learning outcome. Consider, up front, how creating the emotional states can be achieved. On the Microlearning Course Outline form, this can be refined in sections entitled "Focused Outcome" and "How does a proficient individual achieve the outcome now?"

Outline the key messages

Document the key messages, in bullet points, that must be delivered in order for the focused learning outcome to be achieved. Ruthlessly identify all fluff by asking "does the learner need to know/do this to be successful when this lesson is delivered?" and take it out. Make sure you aren't burdening the learner with more information than is absolutely essential. On the Microlearning Course Outline form, this can be refined in sections entitled "What are the key points or messages to be presented?" and "Chunking". These messages will be leveraged as a checklist in subsequent efforts in this workflow.

Select a delivery methodology and channel

Determine the best platform on which to deliver the key messages so that learning is convenient, in line with current habits, and will integrate well with tools and other learning initiatives. There are many ways to deliver microlearning – but having a known tool set that will help you be effective is critical. On the Microlearning Course Outline form, this can be refined in the section entitled "What media could contribute to achieving the outcomes, including emotional states?". The Instructional Designer will also reference selected media in the Editorial Outline for the lesson.

Establish the outline

Working from your key points or messages, evolve step by step content that addresses your key messages, outlines the media to be used, caps the time allowed for the delivery, and indicates any audio narration or multimedia handling. In many cases, this content originates from something much too long that has been provided by Subject Matter Experts – so you may have some back and forth with (an) SME(s) to gain clarity.

Keep a close eye on the time of each key message being delivered. When it runs long, consider whether you are trying to achieve too much in a single lesson against your focused learning outcome. A brief storyboard table can help you document content, media, narrative, and handling. Use expanded storyboards if that is helpful – but beware, typical storyboards can actually prompt you to draw out your content beyond the reasonable intentions of microlearning.

Identify your post-course knowledge translation supports

Microlearning provides a very brief burst of information and activity yet seeks to ensure that the learner either commits the information to memory or, better still, takes some action. Be sure, as you consider the entirety of the microlearning course/lesson, to pinpoint what will support the learner, after the interaction, achieve the outcome. Case studies, email reminders, periodic quizzes, and more can all reinforce after the fact. On the Microlearning Course Outline form, this can be refined in the section entitled "How will you help learners translate knowledge to action or achieve the outcome?"

IMPLEMENTING THE LESSON PLAN

Work with the SMEs

Subject Matter Experts are frequently involved in providing the content for learning. Microlearning is an unfamiliar approach for many, so be ready (either during the Instructional Design stage or as you work towards a lesson build) to answer their questions and do lots of training and teaching.

Implement from your outline (Or Storyboards)

If you have the design well documented, you can efficiently develop in the medium or media selected using your tool set.

Know your tool set

If you are not the implementer of the microlearning lesson, be sure that you at least are familiar with the capabilities of the tool set you will use. Can the tools be used to create video permit interactive branching and on-screen text delivery? What is the range of options in your interactivity builder? Are these tools going to deliver device-agnostic results, or will you be limited to certain screen sizes? Will the tools limit the number of iterations for development efforts?

Compose – and check!

Iterative design is a common notion to most Instructional Designers, but when working with some tools, you need to be aware of limitation. Once that is clear, build, check, then adjust. Normal development processes can apply to microlearning.

Challenge every word

One of the best professional lessons presented to me was to challenge the value of every word. In microlearning, where the time for delivery is very short, and delivery is commonly to smartphones and tablets, every word used onscreen or in an audio narrative is precious. Be relentless in cutting words where pictures will do.

Test, then fix

Again, usual composing practices apply to microlearning. Be sure to test solid alphas and betas with others – particularly, if possible, those in the learning target audience. This testing is not just for technical problems – but for content focus, user experience, and even emotional engagement.

Deploy the call to action

Remember the emphasis, in microlearning, on just one learning outcome per lesson? Here's where everything culminates – when the user hits the program's last screen or page, what is he or she called upon to do? Just as clarity in language and message is sought in microlearning, the learner should have no doubts – and be fully empowered – to act in order for the learning outcome to be fully realized. Borrow here from the marketing playbook on clear call to action.

WHAT ARE THE BENEFITS OF TEACHING MATH USING TECHNOLOGY?

This is a guest blog post from **Kristine Scharaldi**, an education consultant and instructional coach with a specialization in the fields of educational technology, Mind-Brain-Education, Universal Design for Learning (UDL), and 21st Century Skills/Global Education.

Technology provides dynamic opportunities for instruction in math and STEM classrooms. We can enhance the learning process and make concepts come alive through engaging and interactive media. We may also offer additional supports to address the needs of all learners and create customized learning experiences. Here are some important ways that students can benefit when we incorporate technology with our math and STEM lesson instruction.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

TEACHING MATH USING TECHNOLOGY - USE MULTIMEDIA

Multimedia brings learning to life! We can bring videos, animations, interesting movies and other media into the learning process to help our students develop skills and understandings. And it can help to motivate and excite our students about their learning!

Mr. DeMaio, a third grade teacher in Union Beach, New Jersey, creates customized movies to help his students understand class topics such as **multiplication tables** and **borrowing in subtraction**. He hosts a **YouTube channel** with "edu-taining" lessons and music videos that feature teachers in the school and recurring favorite characters such as puppets Steven and Andy.

The movies are so enjoyable to watch that kids play them again and again and ask for more on different topics! Compared to prior school years, Mr. DeMaio has found that this multimedia approach to blended learning has led to better retention and increased student understanding of the concepts, even in math and STEM lessons.

We won't all produce movies like Mr. DeMaio, but he is a good example of how we all have the ability to find and create great content to share with our students through digital tools, platforms, and apps.

INTERACTIVE VISUALIZATIONS AND EXPLORATIONS

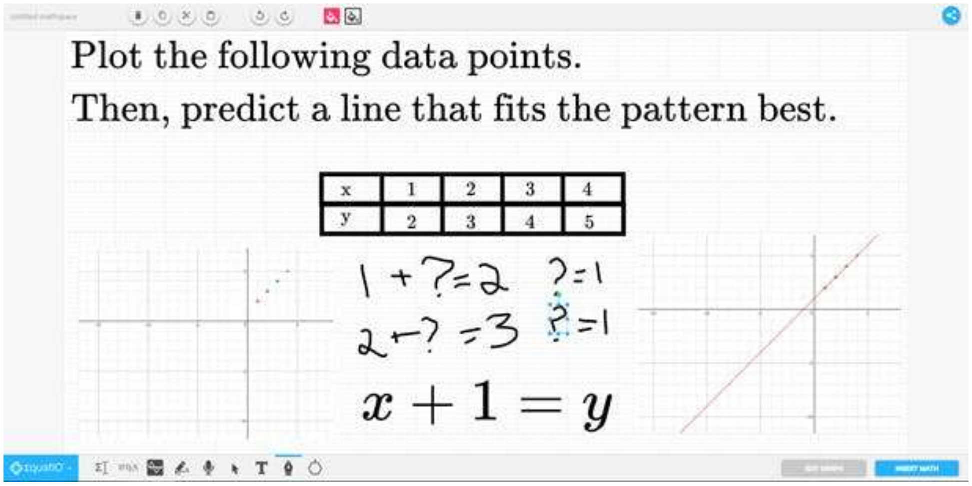
Making math (or STEM subjects) visual goes beyond student engagement; **brain research indicates it to be integral to learning maths**. Neuroscientists at Stanford University are studying how the brain thinks mathematically and evidence shows that visual pathways are involved even when working on symbolic number calculations. According to Jo Boaler and the team at Stanford Graduate School of Education's **youcubed**, representing all mathematical concepts visually, and including visual activities at all grade levels, can greatly help students.

Technology provides additional opportunities for learners to see and interact with mathematical concepts. Students can explore and make discoveries with games, simulations and digital tools.

One excellent platform for teachers and students is the web-based graphing calculator, **Desmos**. The **Desmos classroom activities page** is a great starting point to engage students in playing with and testing mathematical ideas and also sharing and collaborating.

And, the new addition to Texthelp's STEM offering, **EquatIO mathspace**, creates a digital whiteboard where students and teachers can combine math equations and formulas with Desmos graphs, geometrics shapes, manipulatives, and freehand drawings to encourage visual problem solving.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)



PERSONALIZED MATH LESSONS USING TECHNOLOGY

Increased access to technology for math allows for a more customized learning experience. Because no two learners are exactly alike, technology can provide individual students with content and supports that are particularly helpful to their individual needs. Kids can view lessons, tutorials, screencasts, and other instructional media on their own device and at their own pace. So if one student is still confused on a topic, and another is ready for additional challenges, technology can enable each to take the appropriate next step.

A great example of how technology empowers learners is the phenomenon of **Khan Academy**. Sal Khan did not intend to build a non-profit educational organization when he started posting the recordings of his math lessons on YouTube (as he delightfully explains in his TED Talk). He was only trying to help tutor his cousins from afar and didn't see any reason to set the videos to Private mode. From the feedback from his cousins, and then from other people from all over the world who found his videos, he realized how valuable this medium was and the importance of being able to choose, rewind, and control the lessons.

The Khan Academy platform has emerged from his work, giving learners personalized learning experiences in a number of ways. For example, users can take quizzes to see what concepts they have mastered and what they need more practice with. Or students that learn better through written text can access transcripts that accompany the instructional videos. Providing the ability to direct and control learning pathways is a powerful reason to include technology in our own math instruction.

CONNECT MATH CONCEPTS TO THE REAL WORLD

Teachers can use technology to help students see how concepts they are learning in the math or STEM classroom can be applied to everyday life. Instead of giving her students a problem-solving worksheet, educator Jennie Magiera recorded a short video in the dairy aisle of the supermarket, posing the real-world problem of deciding what would be the best deal. She challenged her students to figure out what brand and size of cheese to buy based on the prices and promotions seen on the shelves. Recording videos of scenarios outside of the classroom such as this can be done easily with a smartphone and then shared on YouTube or the class website.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Integrating technology in the math classroom allows students to interact with people outside of the classroom to help broaden their understandings and perspectives about what they are studying. Teachers can set up live interactive video calls with experts on a wide variety of curricular topics using sites such as **Skype in the Classroom** and **Nepris**. One teacher on Nepris posted a request for industry experts to share ways they use math concepts in their daily work, and as a result students were able to virtually meet a playground designer who demonstrated how he uses measurement, multiplication, and more in his decision-making and planning.

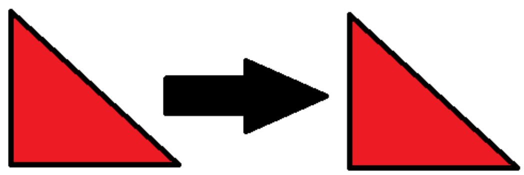
Technology gives us the ability to expand and enrich our math lessons using technology. What ways has technology supported your students in learning math or STEM? Please share your comments!

UNIT 6
RIGID MOTION

RIGID MOTIONS

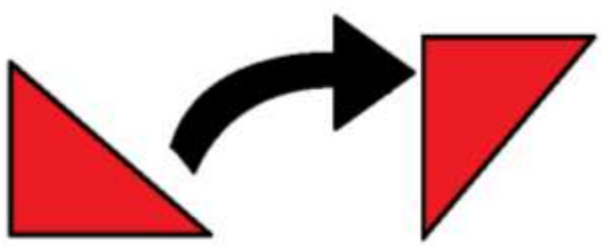
Simply put, rigid motions are transformations that preserve the side lengths and angles in a shape. You can see any of these transformations by moving an object you may have around you right now. Grab your notebook or phone and slide it across the desk you're working on. Now turn it so it's facing sideways. Now flip it over on one of its edges so you're looking at the back side of it. All of these motions are rigid motions.

Let's get into some more formal definitions with some wonderfully drawn MS Paint pictures.



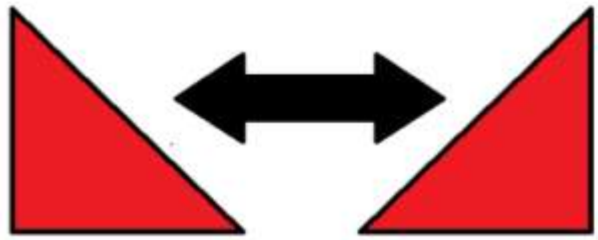
Translation

A **translation** is a transformation that pushes or slides an object. On the picture above, the triangle has just moved to the right a little bit, while keeping its shape. The slide could be in any direction on the plane as long as it keeps its shape, and stays facing the same way.



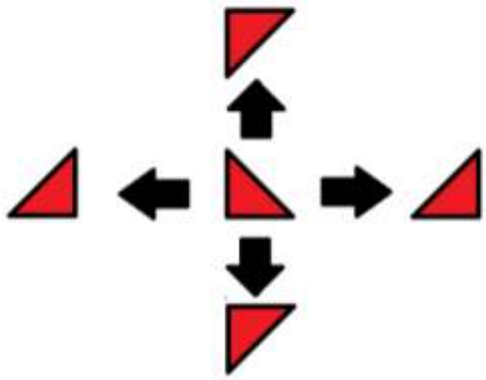
Rotation

A **rotation** is exactly what it sounds like. When the shape turns, you have a rotation. In the picture above, the triangle is turning 90∘ clockwise. You can rotate a shape clockwise or counter clockwise, and for any number of degrees. Once again, the shape is preserved, but now it has a different corner on the bottom, and is facing a different direction. Think of turning a doorknob, when you turn it to one direction, the doorknob still remains the same shape.



Reflection

A **reflection** is when a shape flips. In this case, the triangle has flipped to the right over an imaginary line in the center. You can flip a shape in any direction and over any part of itself, but this can be a little harder to visualize on your own, so here is a picture showing some other types of flips.



Multiple reflections

As you can see, horizontal reflections result in the same shape, and vertical reflections result in the same shape. What has changed between them is what point is being flipped on. For the upwards reflection, the shape is being flipped on the top corner of the triangle, while the downwards reflection is being flipped on the bottom side. The same goes for the horizontal reflections. The right reflection is being flipped on the right most point, and the left reflection is being flipped on the left side.

Let's think about these rigid transformations in a way that is a little big more familiar by using a screen shot from a level in Super Mario Maker.

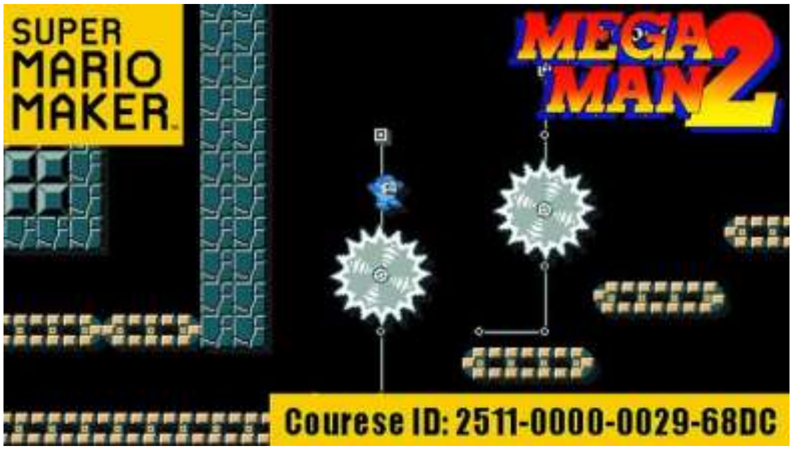


Image by AntMan3001 via Flickr

While this screenshot doesn't have many objects in it, they can demonstrate every rigid transformation.

Image by AntMan3001 via Flickr

The arrows here show the translations within the stage. The saws can move up and down on their tracks, while your character can move in any direction that you want them to. While they move, they retain their shape, otherwise the game wouldn't work as intended. The only slight difference here is that your character may have a slightly different sprite while jumping, but that can be ignored for the sake of visualizing the translations.

Image by AntMan3001 via Flickr

The rotations are shown in this level by the saws. While the saws move up and down on their tracks, they rotate along their center, showing the player that they will in fact cut you up if you land on them.

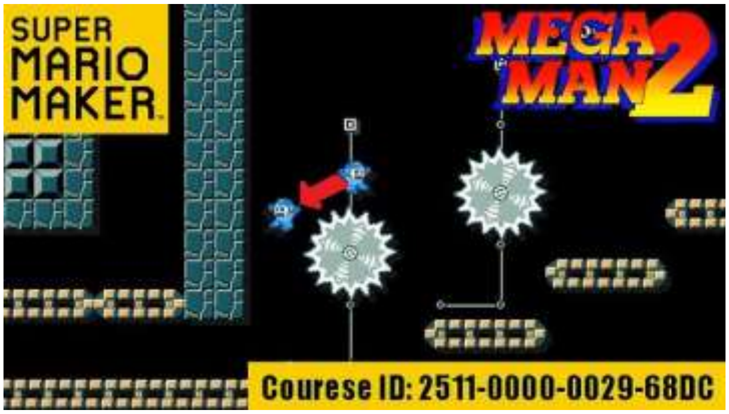


Image by AntMan3001 via Flickr

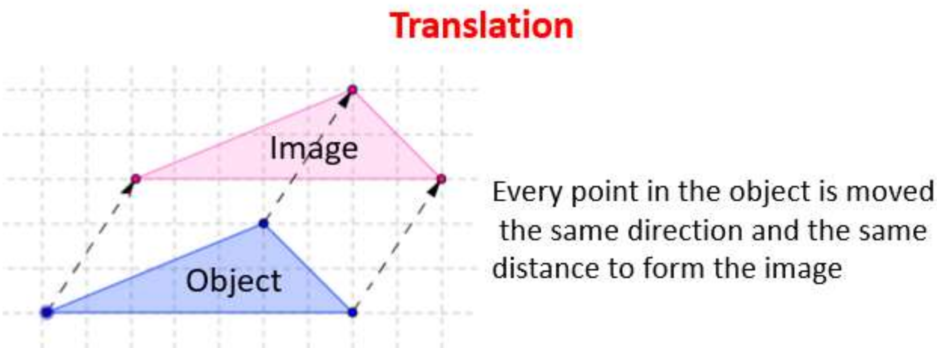
Each time your character moves right or left, they will always face the direction they are moving. In the above picture, let's say you notice that you're going to hit the saw, so you choose to start moving back towards the platform that you jumped from. Your character flips horizontally so they are facing the direction they are moving, while still staying the same exact size and the same position they were in before moving.

To review, the rigid motions are **translations** (slides), **rotations** (spins/turns), and **reflections** (flips). All of these types of motions occur without changing the shape of the object or figure being moved.

TRANSLATION

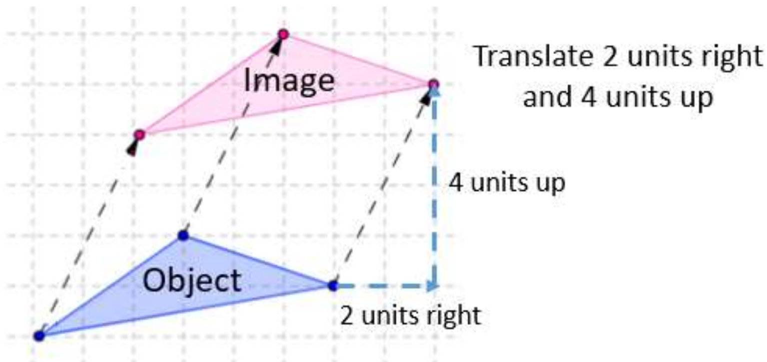
In a **translation transformation** all the points in the object are moved in a straight line in the same direction. The size, the shape and the orientation of the image are the same as that of the original object. Same orientation means that the object and image are facing the same direction.

Example:



We describe a translation in terms of the number of units moved to the right or left and the number of units moved up or down.

Example:
Move the object 2 units to the right and 4 units up.



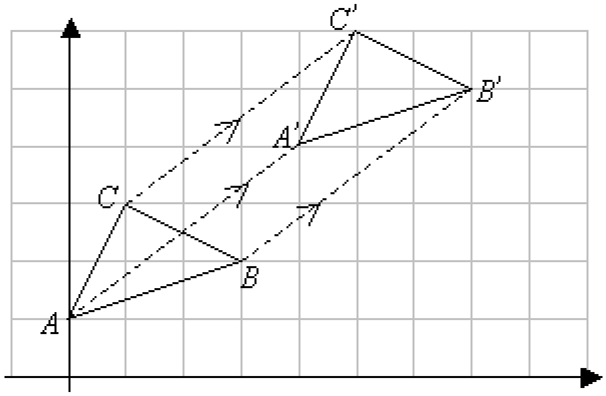
TRANSLATION REPRESENTED BY COLUMN VECTOR OR MATRIX

The translation can be represented by a column vector as $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$.

The top number represents the right and left movement. A positive number means moving to the right and a negative number means moving to the left.

The bottom number represents up and down movement. A positive number means moving up and a negative number means moving down.

In the following figure, triangle ABC is being translated to triangle A'B'C'



The translation is represented by the column vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

In general, a translation can be represented by a **column matrix or column vector** $\begin{pmatrix} a \\ b \end{pmatrix}$ where a is the number of units to move right or left along the x-axis and b is the number of units to move up or down along the y-axis.

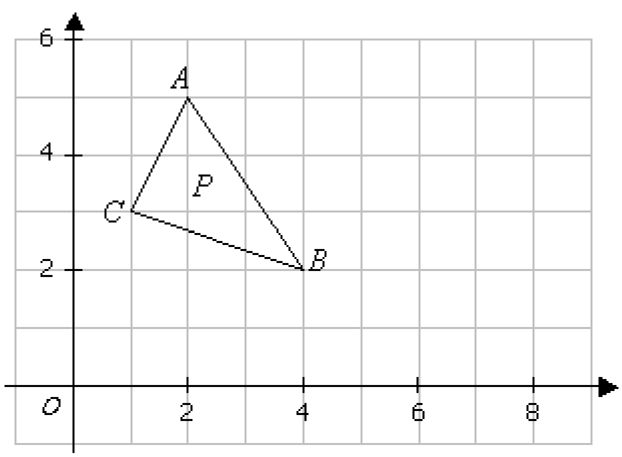
The matrix equation representing a translation is:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}$$

where $\begin{pmatrix} a \\ b \end{pmatrix}$ is the translation matrix and $\begin{pmatrix} x' \\ y' \end{pmatrix}$ is the image of $\begin{pmatrix} x \\ y \end{pmatrix}$.

Example 1:

The triangle P is mapped onto the triangle Q by the translation $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$.



a) Find the coordinates of triangle Q.
b) On the diagram, draw and label triangle Q.

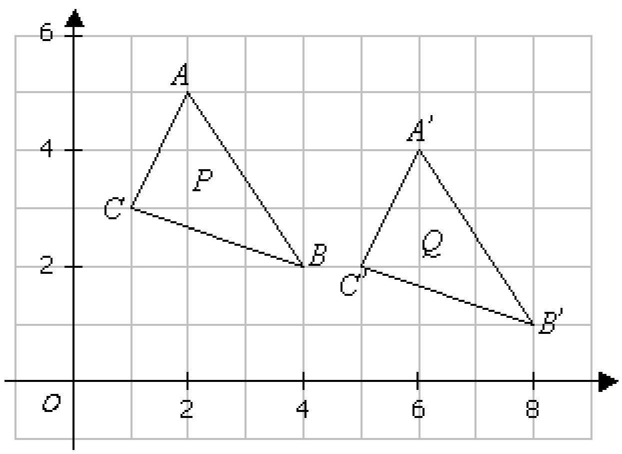
Solution:
a)

$$A' = \begin{pmatrix} 2 \\ 5 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \end{pmatrix}$$

$$B' = \begin{pmatrix} 4 \\ 2 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$$

$$C' = \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$$

b)



As a mathematical notation, we may write: T(A) = B, to mean object A is mapped onto B under the transformation T.

Describing translations of simple shapes in the plane, using column vector notation

- **Show Video Lesson**

How to transform a shape using a given vector?

- **Show Video Lesson**

TRANSLATION ON THE COORDINATE PLANE

Geometry Translation
A geometry translation is an isometric transformation, meaning that the original figure and the image are congruent. Translating a figure can be thought of as "sliding" the original. If the image moved left and down, the rule will be (x - __, y - __) where the blanks are the distances moved

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

along each axis; for translations left and up: (x - __, y + __), for right and down (x + __, y - __), for right and up (x + __, y + __).

- Show Video Lesson

HOW TO TRANSLATE A POLYGON ON THE COORDINATE PLANE?

- Show Video Lesson

Try the free Mathway calculator and problem solver below to practice various math topics. Try the given examples, or type in your own problem and check your answer with the step-by-step explanations.

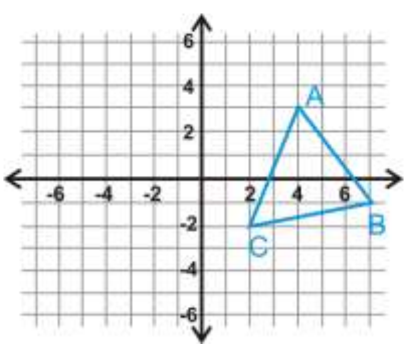
REFLECTIONS OVER AN AXIS

The next transformation is a reflection. Another way to describe a reflection is a "flip."

Reflection: A transformation that turns a figure into its mirror image by flipping it over a line.

Line of Reflection: The line that a figure is reflected over.

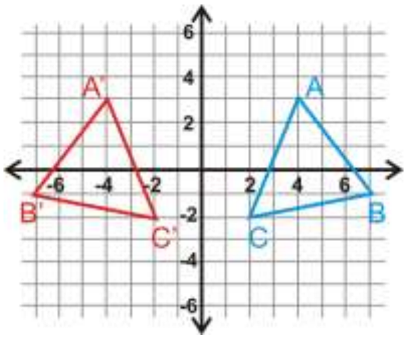
Example 1: Reflect △ABC over the y−axis. Find the coordinates of the image.



[Figure 2]

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Solution: To reflect △ABC over the y−axis the y−coordinates will remain the same. The x−coordinates will be the same distance away from the y−axis, but on the other side of the y−axis.

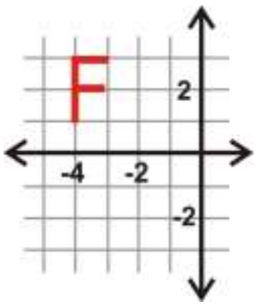


[Figure 3]

A(4,3)B(7,−1)C(2,−2)→A′(−4,3)→B′(−7,−1)→C′(−2,−2)

From this example, we can generalize a rule for reflecting a figure over the y−axis.

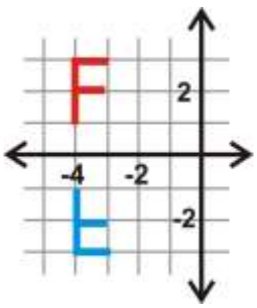
Reflection over the y−**axis:** If (x,y) is reflected over the y−axis, then the image is (−x,y).



[Figure 4]

Example 2: Reflect the letter "F" over the x−axis.

Solution: To reflect the letter F over the x−axis, now the x−coordinates will remain the same and the y−coordinates will be the same distance away from the x−axis on the other side.



[Figure 5]

The generalized rule for reflecting a figure over the x−axis:

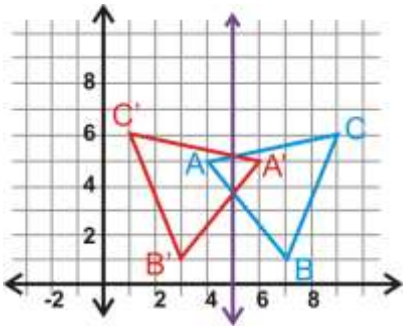
Reflection over the x−axis: If (x,y) is reflected over the x−axis, then the image is (x,−y).

REFLECTIONS OVER HORIZONTAL AND VERTICAL LINES

Other than the x and y axes, we can reflect a figure over any vertical or horizontal line.

Example 3: Reflect the triangle △ABC with vertices A(4,5),B(7,1) and C(9,6) over the line x=5.

Solution: Notice that this vertical line is through our preimage. Therefore, the image's vertices are the same distance away from x=5 as the preimage. As with reflecting over the y−axis (or x=0), the y−coordinates will stay the same.



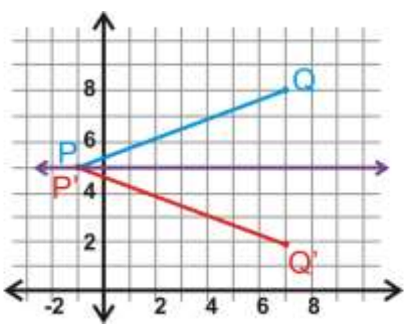
[Figure 6]

A(4,5)B(7,1)C(9,6)→A′(6,5)→B′(3,1)→C′(1,6)

Example 4: Reflect the line segment $\overline{PQ}$ with endpoints P(−1,5) and Q(7,8) over the line y=5.

Solution: Here, the line of reflection is on P, which means P′ has the same coordinates. Q′ has the same x−coordinate as Q and is the same distance away from y=5, but on the other side.

P(−1,5)Q(7,8)→P′(−1,5)→Q′(7,2)



[Figure 7]

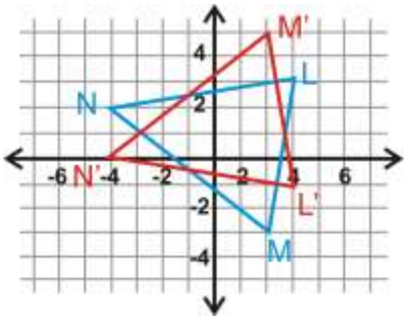
Reflection over x=a: If (x,y) is reflected over the vertical line x=a, then the image is (2a−x,y).

Reflection over y=b: If (x,y) is reflected over the horizontal line y=b, then the image is (x,2b−y).

From these examples we also learned that if a point is on the line of reflection then the image is the same as the original point.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Example 5: A triangle △LMN and its reflection, △L′M′N′ are to the left. What is the line of reflection?



[Figure 8]

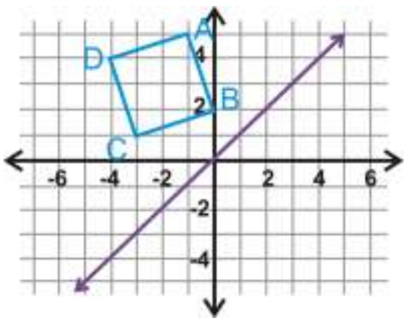
Solution: Looking at the graph, we see that the preimage and image intersect when y=1. Therefore, this is the line of reflection.

If the image does not intersect the preimage, find the midpoint between a preimage and its image. This point is on the line of reflection. You will need to determine if the line is vertical or horizontal.

Reflections over y=x **and** y=−x

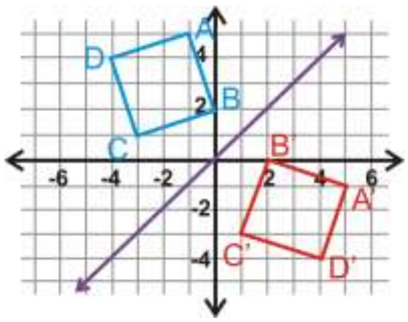
Technically, any line can be a line of reflection. We are going to study two more cases of reflections, reflecting over y=x and over y=−x.

Example 6: Reflect square ABCD over the line y=x.



[Figure 9]

Solution: The purple line is y=x. To reflect an image over a line that is not vertical or horizontal, you can fold the graph on the line of reflection.



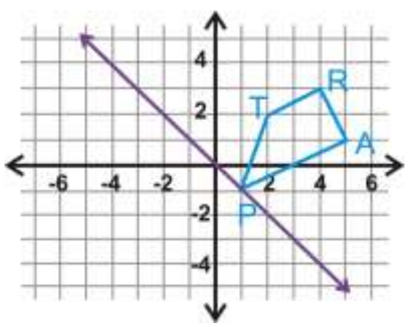
[Figure 10]

A(−1,5)B(0,2)C(−3,1)D(−4,4)→A′(5,−1)→B′(2,0)→C′(1,−3)→D′(4,−4)

From this example, we see that the x and y values are switched when a figure is reflected over the line y=x.

Reflection over y=x: If (x,y) is reflected over the line y=x, then the image is (y,x).

Example 7: Reflect the trapezoid *TRAP* over the line y=−x.

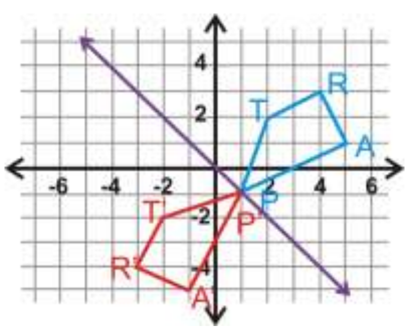


[Figure 11]

Solution: The purple line is y=−x. You can reflect the trapezoid over this line just like we did in Example 6.

T(2,2)R(4,3)A(5,1)P(1,−1)→T′(−2,−2)→R′(−3,−4)→A′(−1,−5)→P′(1,−1)

From this example, we see that the x and y values are switched and the signs are changed when a figure is reflected over the line y=x.



[Figure 12]

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Reflection over y=−x: If (x,y) is reflected over the line y=−x, then the image is (−y,−x).

At first glance, it does not look like P and P′ follow the rule above. However, when you switch 1 and -1 you would have (-1, 1). Then, take the opposite sign of both, (1, -1). Therefore, when a point is on the line of reflection, it will be its own reflection.

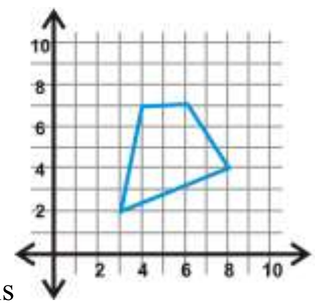
From all of these examples, we notice that ***a reflection is an isometry***.

Know What? Revisited The white line in the picture is the line of reflection. This line coincides with the water's edge. If we were to place this picture on the coordinate plane, the line of reflection would be any horizontal line. One example could be the x−axis.

Review Questions

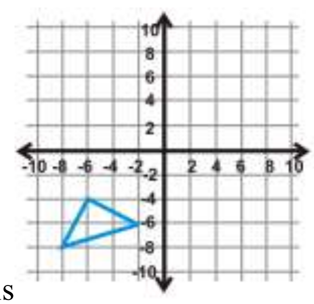
1. Which letter is a reflection over a vertical line of the letter "b"?
2. Which letter is a reflection over a horizontal line of the letter "b"?

Reflect each shape over the given line.



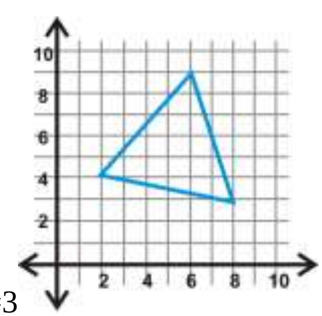
3. y−axis

[Figure 14]



4. x−axis

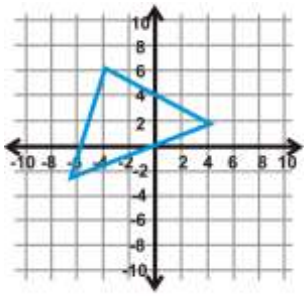
[Figure 15]



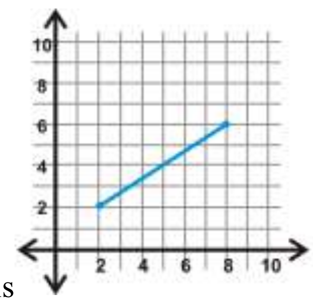
5. y=3

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

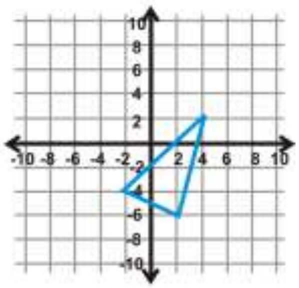
[Figure 16]

6. x=−1 

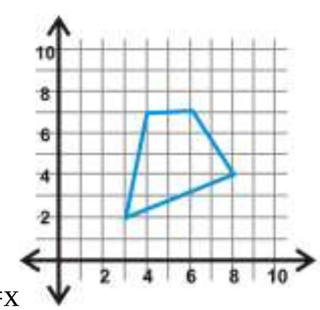
[Figure 17]

7. x−axis 

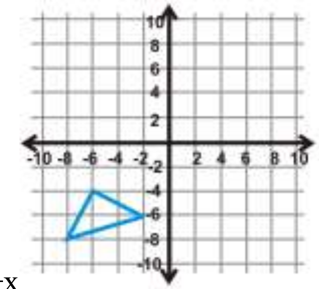
[Figure 18]

8. y−axis 

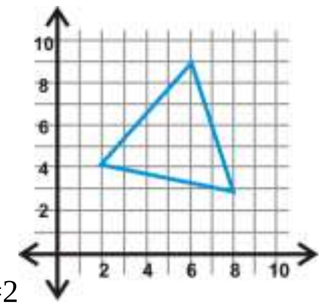
[Figure 19]

9. y=x 

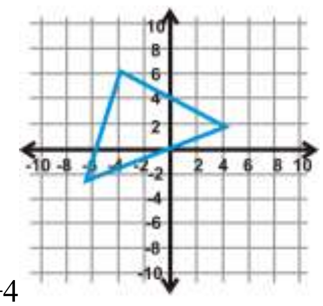
[Figure 20]


10. y=−x

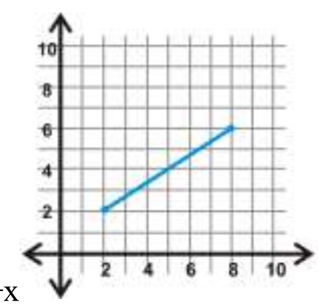
[Figure 21]


11. x=2

[Figure 22]

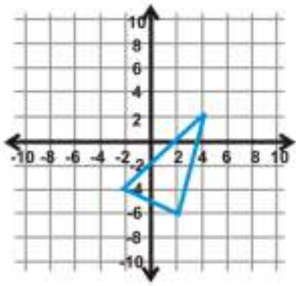

12. y=−4

[Figure 23]


13. y=−x

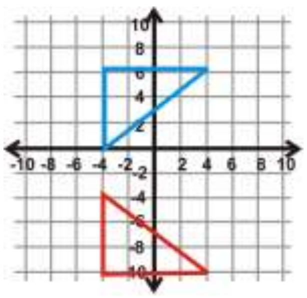
[Figure 24]

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

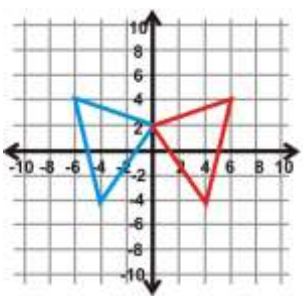
14. y=x 

[Figure 25]

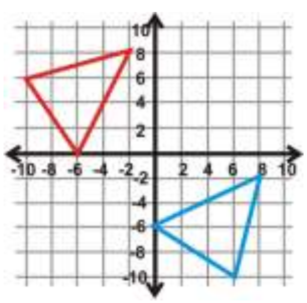
Find the line of reflection of the blue triangle (preimage) and the red triangle (image).

15. 

[Figure 26]

16. 

[Figure 27]

17. 

[Figure 28]

Two Reflections The vertices of △ABC are A(−5,1),B(−3,6), and C(2,3). Use this information to answer questions 18-21.

18. Plot △ABC on the coordinate plane.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

19. Reflect $\triangle ABC$ over $y=1$. Find the coordinates of $\triangle A'B'C'$.

20. Reflect $\triangle A'B'C'$ over $y=-3$. Find the coordinates of $\triangle A''B''C''$.

21. What ***one*** transformation would be the same as this double reflection?

Two Reflections The vertices of $\triangle DEF$ are $D(6,-2)$,$E(8,-4)$, and $F(3,-7)$. Use this information to answer questions 22-25.

22. Plot $\triangle DEF$ on the coordinate plane.

23. Reflect $\triangle DEF$ over $x=2$. Find the coordinates of $\triangle D'E'F'$.

24. Reflect $\triangle D'E'F'$ over $x=-4$. Find the coordinates of $\triangle D''E''F''$.

25. What ***one*** transformation would be the same as this double reflection?

Two Reflections The vertices of $\triangle GHI$ are $G(1,1)$,$H(5,1)$, and $I(5,4)$. Use this information to answer questions 26-29.

26. Plot $\triangle GHI$ on the coordinate plane.

27. Reflect $\triangle GHI$ over the $x-$axis. Find the coordinates of $\triangle G'H'I'$.

28. Reflect $\triangle G'H'I'$ over the $y-$axis. Find the coordinates of $\triangle G''H''I''$.

29. What ***one*** transformation would be the same as this double reflection?

30. Following the steps to reflect a triangle using a compass and straightedge.

    a. Make a triangle on a piece of paper. Label the vertices A,B and C.

    b. Make a line next to your triangle (this will be your line of reflection).

    c. Construct perpendiculars from each vertex of your triangle through the line of reflection.

    d. Use your compass to mark off points on the other side of the line that are the same distance from the line as the original A,B and C. Label the points A',B' and C'.

    e. Connect the new points to make the image $\triangle A'B'C'$.

31. Describe the relationship between the line of reflection and the segments connecting the preimage and image points.

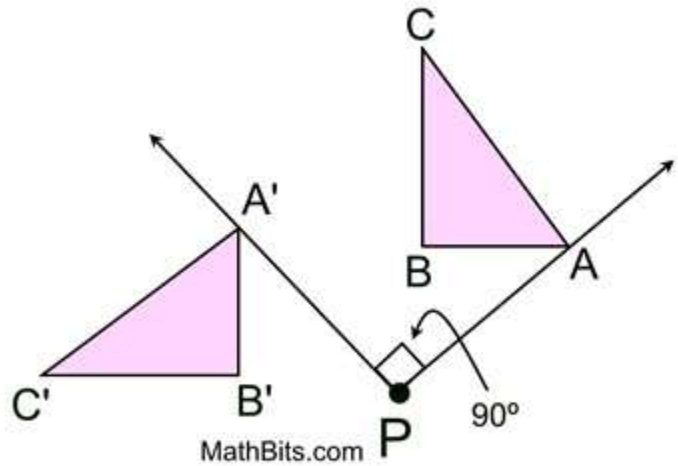
32. Repeat the steps from problem 28 with a line of reflection that passes ***through*** the triangle.

ROTATIONS

A rotation is a transformation that turns a figure about a fixed point called the center of rotation.
• An object and its rotation are the same shape and size, but the figures may be turned in different directions.
• Rotations may be clockwise or counterclockwise.

When working in the coordinate plane:
• assume the center of rotation to be the origin unless told otherwise.
• assume a positive angle of rotation turns the figure counterclockwise, and a negative angle turns the figure clockwise (unless told otherwise).

com



The triangle is rotated. The letters used to label the triangle have not been rotated.

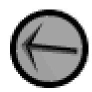
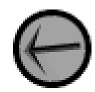
Rotations can be seen, in a variety of situations:

The Earth	Windmills	Pinwheel
The Earth experiences one complete rotation on its axis every 24 hours. 	The blades on windmills convert the energy of wind into rotational energy. 	A children's toy that rotates when blown. 
Amusement Park Swing	Ferris Wheel	Merry-Go-Round

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

An amusement park rides, such as the swing, allow you to become the of the rotation. 	Ferris wheels rotate about a center hub. (Yes, the seats tilt to prevent falling.) 	On the merry-go-round, riders become part of the rotation about the center of the ride. 

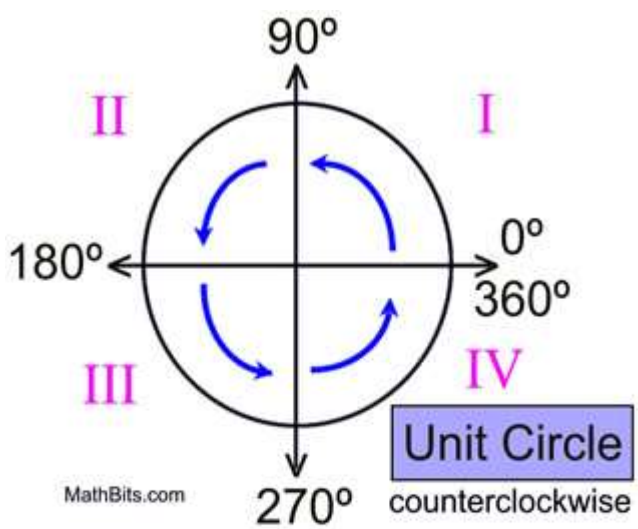
Rotation of 90º:	(*x*,*y*) becomes (-*y*,*x*)
Rotation of 180º:	(*x*,*y*) becomes (-*x*,-*y*)
Rotation of 270º:	(*x*,*y*) becomes (*y*,-*x*)

Remember :	Clockwise: 	Counterclockwise : 

Rotations in the coordinate plane are counterclockwise.

When working with rotations, you should be able to recognize angles of certain sizes. Popular angles include 30º (one third of a right angle), 45º (half of a right angle), 90º (a right angle), 180º, 270º and 360º.

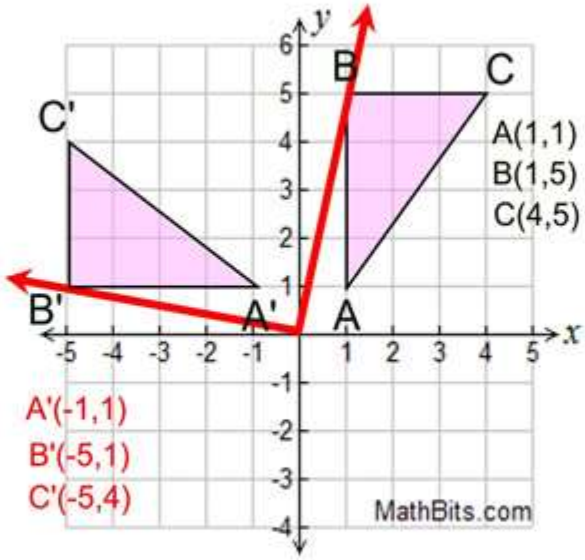
You should also understand the directionality of a unit circle (a circle with a radius length of 1 unit). Notice that the degree movement on a unit circle goes in a counterclockwise direction, the same direction as the numbering of the quadrants: I, II, III, IV. Keep this picture in mind when working with rotations on a coordinate grid.



CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

- Rotation 90º:

Starting with ΔABC, draw the rotation of 90º. (It is assumed that the center of the rotation is the origin and that the rotation is counterclockwise.)



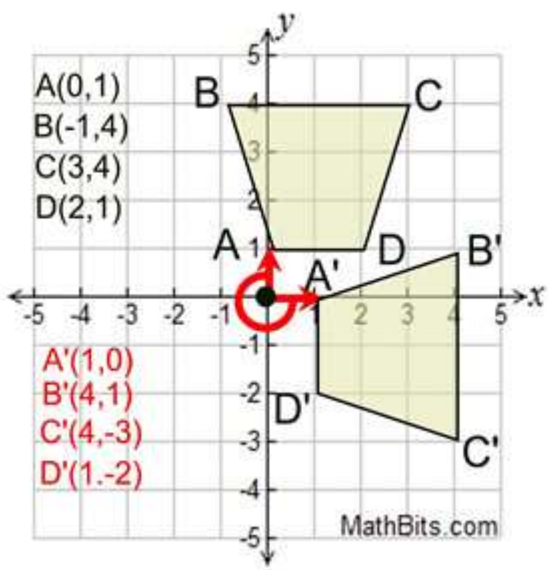
To "see" that this is a rotation of 90º, imagine point *B* attached to the red arrow. The red arrow is then moved 90º (notice the 90º angle formed by the two red arrows). Look at the new position of point *B*, labeled *B'*. This same approach can be used for all three vertices.

Rotation of 90º on coordinate axes.
$(x, y) \rightarrow (-y, x)$

Rotations in the coordinate plane:
Keep in mind that rotations on a coordinate grid are considered to be counterclockwise, unless otherwise stated.

- Rotation 270º:

Starting with quadrilateral *ABCD*, draw the rotation of 270º. (It is assumed that the center of the rotation is the origin and that the rotation is counterclockwise.)



As we did in the previous examples, imagine point *A* attached to the red arrow from the center (0,0). The arrow is then moved 270º (counterclockwise). Notice the new position of *A*, labeled *A'*. Since *A* was "on" the axis, *A*' is also on the axis.

Rotation of 270º on coordinate axes.
$(x, y) \rightarrow (y, -x)$

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

UNIT 7
ALGEBRA

WHAT IS AN ALGEBRAIC EXPRESSION?

An algebraic expression in mathematics is an expression which is made up of variables and constants, along with algebraic operations (addition, subtraction, etc.). Expressions are made up of terms. Also, solve questions in Algebraic Expressions Worksheets, at BYJU'S.

Examples

3x + 4y – 7, 4x – 10, etc.

These expressions are represented with the help of unknown variables, constants and coefficients. The combination of these three (as terms) is said to be an expression. It is to be noted that, unlike the algebraic equation, an algebraic expression has no sides or equal to sign. Some of its examples include

- 3x + 2y – 5
- x – 20
- 2x2 − 3xy + 5

Variables, Coefficient & Constant

In Algebra we work with Variable, Symbols or Letters whose value is unknown to us.

Example

5x – 3

★ **x** is a **variable**; whose value is unknown to us which can take any value.

★ **5** is known as the **coefficient** of x, as it is a constant value used with the variable term and is well defined.

★ **3** is the **constant** value term which has a definite value.

The whole expression is known to be the Binomial term, as it has two unlikely terms.

<u>Types of Algebraic expression</u>

There are 3 main types of algebraic expressions which include:

- Monomial Expression
- Binomial Expression
- Polynomial Expression

- Monomial Expression

An algebraic expression which is having only one term is known as a monomial.

Examples of monomial expression include 3x4, 3xy, 3x, 8y, etc.

Binomial Expression

A binomial expression is an algebraic expression which is having two terms, which are unlike.

Examples of binomial include 5xy + 8, xyz + x3, etc.

Polynomial Expression

In general, an expression with more than one terms with non-negative integral exponents of a variable is known as a polynomial.

Examples of polynomial expression include ax + by + ca, x3 + 2x + 3, etc.

Other Types of Expression:

Apart from monomial, binomial and polynomial types of expressions, an algebraic expression can also be classified into two additional types which are:

- Numeric Expression
- Variable Expression

Numeric Expression

A numeric expression consists of numbers and operations, but never include any variable. Some of the examples of numeric expressions are 10 + 5, 15 ÷ 2, etc.

Variable Expression

A variable expression is an expression which contains variables along with numbers and operation to define an expression. A few examples of a variable expression include 4x + y, 5ab + 33, etc.

How to Simplify Algebraic Expressions

To simplify an algebraic expression, we just combine the like terms. Hence, the like variables will be combined together. Now, out of the like variables, the same powers will be combined together. For example, let us take an algebraic expression and try to reduce it to its lowest form in order to understand the concept better. Let our expression be:

x3 + 3x2 − 2x3 + 2x − x2 + 3 − x

= (x3 − 2x3) + (3x2 − x2) + (2x − x) + 3

= −x3 + 2x2 + x + 3

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Hence, the algebraic expression x3 + 3x2 − 2x3 + 2x − x2 + 3 − x simplifies to −x3 + 2x2 + x + 3.

EXPRESSION FORMULAS

Algebraic formulas are the derived short formulas that help us in solving the equations easily. They are just a rearrangement of the given terms in order to create a better expression that is easy to memorize. Find below a list of some of the basic formulas that are being used widely. Have a look at this page in order to understand the algebraic formulas better.

The general algebraic formulas we use to solve the expressions or equations are:

- (a + b) = a2 + 2ab + b2
- (a - b) = a2 - 2ab + b2
- (a + b)(a - b) = a2 - b2
- (x + a)(x + b) = x2 + x(a + b) + ab

LIKE TERM IN ALGEBRA

Like terms are terms with the same combination of letters (and/or brackets). The only difference is the sign or number in front of the group of letters. Each letter (and/or bracket) in a like term must have the same exponents - the number that sits to the top-right of the letter.

TERM IN ALGEBRA

A term is a collection of numbers, letters and brackets all multiplied together. Terms are separated by + or - signs in an algebraic expression.

HOW TO IDENTIFY LIKE TERMS IN ALGEBRA

Identifying like terms is easy. Look for terms that have the same collection of letters, with the same exponent (or power) next to them. It is useful to know how to check for like terms:

- Check that the same variables (letters) appear in each of the terms.
- Check that the exponent of each variable is the same in each of the terms.
- Check that the only difference between the like terms is the coefficient. This is normally the number in front of the term, although a coefficient can sometimes be a letter (a constant).

Question

Identify the like terms in the expression below.

In this example, we can see that there are **x** terms and **x^2y** terms.

x terms

The **x** and **3x** terms look like they are like terms.

Let's check the **x** terms:

Step-by-Step:

1

Check that the same variables appear in both terms.

x 3**x**

The variable **x** appears in both terms.

2

Check that the exponent of each variable is the same in each of the terms.

$x = x^{1}$ $3x = 3x^{1}$

There is no exponent by each **x** in each of the terms. (Actually, the exponent is **1** but there is no need to write it).

3

Check that the only difference between the like terms is the coefficient.

x = **1**x **3x**

There is no coefficient in front of the **x** term. (Actually, the coefficient is **1** but there is no need to write it). The coefficient of the **3x** term is **3**. This is the only difference between each of the like terms.

Answer:

x and **3x** are like terms.

x^2y terms

The **$3x^2y$** and **$-½x^2y$** terms look like they are like terms.

Let's check the **x^2y** terms:

Step-by-Step:

1

Check that the same variables appear in both terms.

$3x^2y$ $-½x^2y$

The variable **x** and **y** appear in both terms.

2

Check that the exponent of each variable is the same in each of the terms.

$3x^2y$ = _$3x^2y^1$ $-½x^2y$ = $-½x^2y^1$

The **x** has an exponent of **2** in each of the terms. There is no exponent for the **y** in each of the terms. (Actually, the exponent is **1** but there is no need to write it).

3

Check that the only difference between the like terms is the coefficient.

3x^2y **−½**x^2y

The coefficient of the $\mathbf{3x^2y}$ term is **3**. The coefficient of the $\mathbf{-½x^2y}$ term is **−½**. This is the only difference between each of the like terms.

Answer:

$\mathbf{3x^2y}$ and $\mathbf{-½x^2y}$ are like terms.

FINDING THE SUM AND DIFFERENCE OF THE SAME TWO TERMS

When distributing binomials over other terms, knowing how to find the sum and difference of the same two terms is a handy shortcut. The sum of any two terms multiplied by the difference of the same two terms is easy to find and even easier to work out — the result is simply the square of the two terms. The middle term just disappears because a term and its opposite are always in the middle.

If you encounter the same two terms and just the sign between them changes, rest assured that the result is the square of those two terms. The second term will always be negative, as in the example,

$$(a+b)(a-b) = a^2 - b^2$$

Example 1: $(x - 4)(x + 4)$

You can use the shortcut to do these special distributions.

The first term (x) squared is x^2.

The second term will always be negative, and a perfect square like the first term: (–4)(+4) = –16.

$$\text{So } (x-4)(x+4) = x^2 - 16$$

Example 2: $(ab - 5)(ab + 5)$

Try the same easy process — multiplying the sum of two terms with their difference — with this slightly more complicated, variable term.

$$\text{The square of } ab = (ab)^2 = a^2b^2$$

The second term is negative, and a perfect square like the first term: 5 = –25.

$$\text{So } (ab-5)(ab+5) \;=\; a^2b^2 - 25$$

Example 3: [5 + (*a* – *b*)][5 – (*a* – *b*)]

This example offers you a chance to work through the sum and difference of various groupings.

The square of 5 = 25

The second term is negative, and a perfect square like the first term:

$$(a-b) \;=\; -(a-b)^2$$

Square the binomial and distribute the negative sign, which looks like this:

$$-(a^2 - 2ab + b^2) \;=\; -a^2 + 2ab - b^2$$

$$\text{So } [5+(a-b)][5-(a-b)] \;=\; 25 - a^2 + 2ab - b^2$$

MULTIPLYING BINOMIAL EXPRESSIONS

by Ron Kurtus

A ***binomial expression*** is an algebraic expression consisting of two terms or monomials separated by a plus (+) or minus (−) sign. Examples of binomials include: **ax + b**, $\mathbf{x^2 - y^2}$, and **2x + 3y**.

In Algebra, you are often required to multiply expressions together. The best way to multiply two binomial expressions is to use what is called the **FOIL** method. In this method, you multiply the **F**irst, **O**utside, **I**nside, and **L**ast terms and then add them together.

You should follow some good practices, such as putting terms in the right order, before you start. Also, there are a few shortcuts that are good to remember.

Questions you may have include:

- What is the FOIL method?
- What are some good practices?

- What are some special situations?

This lesson will answer those questions.

FOIL METHOD

Since binomials are simple and you are smart, a **FOIL** method is usually used to multiply two binomials.

FOIL stands for: "multiply the **First** terms, multiply the **Outside** terms, multiply the **Inside** terms, and multiply the **Last** terms." Then you add the results together in the proper order.

For example, to multiply **(ax + b)(cx + d)**, you follow the procedure:

1. Multiply the two **First** terms together **(ax + b)(cx + d)**: **(ax)(cx)** = acx^2

2. Multiply the two **Outside** terms **(ax + b)(cx + d)**: **(ax)(d) = adx**

3. Multiply the two **Inside** terms **(ax + b)(cx + d)**: **(b)(cx) = bcx**

4. Multiply the two **Last** terms **(ax + b)(cx + d)**: **(b)(d) = bd**

5. Add the results to get: $acx^2 + adx + bcx + bd$

This can also be written as: $acx^2 + (ad + bc)x + bd$

Typically, you can do these operations in your head, writing down the results in their order.

Good practices

There are different situations you can study.

Arrange terms

It is a good practice to arrange the terms in a uniform order. For example, supposed you want to multiply **(3y + x)(2x − 5y)**. Although you will get the same answer using that order, it is best and easier to rearrange the terms as **(x + 3y)(2x − 5y)**.

Then multiply **(x + 3y)(2x − 5y)**:

1. First terms: **(x)(2x)** = $2x^2$

2.Outside terms: **(x)(−5y) = −5xy**

3. Inside terms: **(3y)(2x) = 6xy**

4. Last terms: **(3y)(−5y)** = $-15y^2$

5. Add together: $2x^2 - 5xy + 6xy - 15y^2$

6. Combine like terms to get the final result: $2x^2 + xy - 15y^2$

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Likewise, you should rearrange **(xa − 7)(5x + 2)** to be **(ax − 7)(5x + 2)** before multiplying.

Simplify expressions

You also want to simplify expressions, if possible. For example, the expression **2x + 3 − 8** can be simplified into a binomial expression: **2x −5**.

Special situations

There are special situations to be aware of.

When binomials not similar

When the binomials are not similar, it can get tricky. For example, multiply $(x^2 - y)(x - 2y)$:

1. First: $(x^2)(x) = x^3$

2.Outside: $(x^2)(-2y) = -2x^2y$

3. Inside: $(-y)(x) = -xy$ (note that we changed the order of **x** and **y**)

4. Last: $(-y)(-2y) = 2y^2$

5. Add together: $x^3 - 2x^2y - xy + 2y^2$

This result has four terms instead of the usual three terms.

Squaring an expression

If you are going to multiply an expression by itself, such as $(ax + b)^2 = (ax + b)(ax + b)$, it is easy to get the result without the **FOIL** method.

$(ax + b)(ax + b) = a^2x^2 + 2abx + b^2$

Also

$(ax - b)(ax - b) = a^2x^2 - 2abx + b^2$

It is good to remember this shortcut.

Special form

When you multiply binomials in the form of **(a + b)(a −b)**, the result is $a^2 - b^2$.

For example, multiply **(2x + 3y)(2x − 3y)**

1. First: $(2x)(2x) = 4x^2$

2. Outside: $(2x)(-3y) = -6xy$

3. Inside: $(3y)(2x) = 6xy$

4. Last: $(3y)(-3y) = -9y^2$

5. Add together: $4x^2 - 6xy + 6xy - 9y^2$ for the final result:

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

$\mathbf{4x^2 - 9y^2}$ or $\mathbf{2^2x^2 - 3^2y^2}$

Remember this shortcut, because it will come up time and time again in Algebra.

Another example is simply: $\mathbf{(x - 2)(x + 2) = x^2 - 4}$

Exercises

Try the following exercises:

1. $\mathbf{(5x - 7)(x + 2)}$
2. $\mathbf{(x^2 + 3)(x^2 + 3)}$
3. $\mathbf{(x - y)(3x - 2y)}$
4. $\mathbf{(4 - y)(y + 4)}$
5. $\mathbf{(a + b)(c + d)}$

Answers

1. $\mathbf{5x^2 + 3x - 14}$
2. $\mathbf{x^4 + 6x^2 + 9}$
3. $\mathbf{3x^2 - 5xy + 2y^2}$
4. $\mathbf{16 - y^2}$
5. $\mathbf{ac + ad + bc + bd}$

Summary

You are often required to multiply expressions together. The best way to multiply two binomial expressions is to use what is called the **FOIL** method.

There are some good practices to follow, such as putting terms in the right order.

Some shortcuts to remember are:

$\mathbf{(a + b)(a + b) = a^2 + 2ab + b^2}$

$\mathbf{(a - b)(a - b) = a^2 - 2ab + b^2}$

$\mathbf{(a + b)(a - b) = a^2 - b^2}$

FACTORING BINOMIALS: HOW TO BREAK DOWN ALGEBRAIC EXPRESSIONS

Factoring binomials means breaking down a binomial expression into its simplest form. Binomials are algebraic expressions with two terms. When factoring binomials, you are required to separate the expression into two simpler expressions surrounded by parentheses:

$$x^2 - 16 \rightarrow (x - 4)(x + 4)$$

In this article, we'll cover how to factor binomials and trinomials.

WHAT ARE BINOMIALS?

All of the following expressions are examples of binomials:

- $2x+3$
- $6x^2-7$
- $-12.3x+9y^2$
- x^3-14

As you can see, any algebraic expression with two terms is a binomial. Binomials can include whole numbers, negative numbers, decimals, or exponents of any value. And it doesn't matter whether the first term or the second term has a variable or not. The only real rule of a binomial is that it has two terms and at least one of them has a variable such as x.

A GUIDE TO FACTORING BINOMIALS

Here is an example of a factorable binomial:

x^2-9

The algebraic expression above is an example of a binomial that can be factored, or put in its simplest form because you can take the square root of both x^2 and 9.

To factor binomials with exponents to the second power, take the square root of the first term *and* of the coefficient that follows. We'll look at each part of the binomial separately.

$\sqrt{x^2}=x$

$\sqrt{9}=3$

In this binomial, you're subtracting 9 from x^2. Factoring a binomial that uses subtraction to split up the square root of a number is called the difference of two squares. Since multiplying a negative by a positive equals a negative number, the factorization of this binomial will have to include a positive *and* a negative 3. When you simplify this binomial, you get this factored form:

$(x-3)(x+3)$

BINOMIALS THAT CAN'T BE FACTORED

Though many different types of expressions can be classified as binomials, not all of them can be factored. In order to be factorable, a binomial has to have a difference of two squares, a difference of cubes, a sum of cubes, or a greatest common factor. We'll explain the latter three terms below.

The following is an example of a non-factorable binomial:

$3x^2+14$

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Here is why this binomial expression cannot be factored:

1. **Unfactorable leading coefficient**: The leading coefficient, which is the number written in front of the variable with the largest exponent, is 3. Since 3 is a prime number whose square or cubed root cannot be taken, you can't break this binomial up into two expressions.
2. **There is no greatest common factor (GCF)**: A GCF is a factor that both terms within the binomial expression have in common. Since there is no common factor between 3 and 14, they can't be divided into two expressions.

DIFFERENCE AND SUM OF CUBES

When a binomial doesn't have a GCF or difference of two squares, it has to have either a difference of cubes or a sum of cubes in order to be factorable. These types of cubed binomial expressions must be written in the following format:

$Difference\ of\ cubes{:}a^3 - b^3$

$Sum\ of\ cubes{:}a^3 + b^3$

In this expression, *a* and *b* represent coefficients. Like any other binomial, this cubed expression has to include at least one variable, such as *x*, to be factorable. The factored form will be separated into a two-term expression and a three-term expression:

$Factored\ form\ of\ the\ difference\ of\ cubes:$

$a^3 - b^3 \rightarrow (a-b)(a^2+ab+b^2)$

$Factored\ form\ of\ the\ sum\ of\ cubes:$

$a^3 + b^3 \rightarrow (a+b)(a^2-ab+b^2)$

Let's use these formulas to factor this difference of cubes expression:

$27x^3-64$

Let's check if coefficients 27 and 64 can be cubed. If so, *a* will equal the cubed root of 27, and *b* will equal the cubed root of 64:

$\sqrt[3]{27} = 3$

$\sqrt[3]{64} = 4$

Since both of these numbers can be cubed, let's plug in the values of *a* and *b* to the factored form for the difference of cubes:

$(3x-4)(3x^2+(3x\times4)+(4)^2)\rightarrow(3x-4)(3x^2+12x+16)$

FACTORING TRINOMIALS

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Binomials are part of a larger group of expressions called polynomials. Other examples of polynomials are monomials, an expression with only one term, and trinomials, an expression with three terms. Here are examples of each:

Monomial: $4xy^2$

Trinomial: $x^2+7x+10$

In order to factor trinomials, you have to find a common factor between all three terms. By splitting up the middle term of the trinomial above into 2x and 5x, you can find the greatest common denominator of the first term x^2+2x and the last term 5x+10, respectively:

$x^2+7x+10$

$x^2+2x+5x+10$

$x(x+2)+5(x+2)$

Take the outside expression of x+5 and the inner expression of x+2 to get the factored form of this trinomial:

$(x+2)(x+5)$

THE ART OF FACTORING BINOMIALS

By simplifying a two-term expression, you're putting your knowledge of greatest common denominators and square roots to work. And though factoring binomials takes practice, mastering it makes it easier to factor more complex polynomial expressions.

WHAT IS A POWER IN MATH?

A **power** is the product of several identical factors. For example:

$2 \times 2 \times 2$

The value of this expression is 8

$2 \times 2 \times 2 = 8$

The left part of this equation can be made shorter - first write the repeating multiplier and specify over it how many times it repeats. The repeating factor in this case is 2. It is repeated three times. So above the 2 we write the 3:

$2^3 = 8$

The expression reads *"two to the third power equals eight"* or *"the third power of 2 equals 8"*.

The short form of recording multiplication of identical factors is used more often. Therefore, remember that if another number is written over some number, it is the multiplication of several identical factors.

For example, if an expression 5^3 is given, keep in mind that this expression is equivalent to writing $5 \times 5 \times 5$.

The number that is repeated is called the **base of the power**. In expression 5^3, the base of the power is number 5.

The number written above number 5 is called the **exponent**. In expression 5^3, the exponent is number 3. The exponent shows how many times the base of the power is repeated. In this case, base 5 is repeated three times

The operation of multiplying identical factors is called **exponentiation**.

For example, if you want to find the product of four identical factors, each of which is equal to 2, they say that the number 2 is **raised to the fourth power**:

We see that the number 2 to the fourth power is number 16.

Note that in this lesson we are looking at **powers with a natural exponent**. This is a type of power whose exponent is a natural number. Recall that natural numbers are integers that are greater than zero. For example, 1, 2, 3, and so on.

In general, the definition of a power with a natural exponent is as follows:

The power of a with a natural exponent n is an expression of the form a^n, which is equal to the product of n factors, each of which is equal to a

examples:

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

$$a^2 = \underbrace{aa}_{\text{2 times}}$$

$$a^3 = \underbrace{aaa}_{\text{3 times}}$$

$$a^4 = \underbrace{aaaa}_{\text{4 times}}$$

$$a^5 = \underbrace{aaaaa}_{\text{5 times}}$$

One must be careful when taking a number to the power. Often a person is inattentive and multiplies the base of a power by the exponent.

For example, the number 5 to the second power is the product of two factors, each of which is 5. This product is 25

$$5^2 = 5 \times 5 = 25$$

Now suppose we inadvertently multiplied base 5 by 2.

$$5^2 \neq 5 \times 2 = 10$$

There is an error because the number 5 to the second power is not equal to 10.

Additionally, it should be mentioned that the power of a number with exponent 1 is the number itself:

$$a^1 = a$$

For example, the number 5 to the first power is the number 5 itself

$$5^1 = 5$$

Accordingly, if a number has no exponent, then we must assume that the exponent is equal to one.

For example, numbers 1, 2, 3 are given without exponent, so their exponents will be equal to one. Each of these numbers can be written with exponent 1

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

$1 = 1^1$

$2 = 2^1$

$3 = 3^1$

And if you exponentiate 0, you get 0. Indeed, no matter how many times nothing is multiplied by itself you get nothing. Examples:

$0^1 = 0$

$0^2 = 0 \times 0 = 0$

$0^3 = 0 \times 0 \times 0 = 0$

And expression 0^0 has no meaning. But in some branches of mathematics, in particular analysis and set theory, expression 0^0 may make sense.

To practice, let's solve some examples on raising numbers to a power.

Example 1. To raise number 3 to the second power.

Number 3 to the second power is the product of two factors, each of which is equal to 3

$3^2 = 3 \times 3 = 9$

Example 2. To raise the number 2 to the fourth power.

The number 2 to the fourth power is the product of four factors, each of which is 2

$2^4 = 2 \times 2 \times 2 \times 2 = 16$

Example 3. Raise number 2 to the third power.

The number 2 to the third power is the product of three factors, each of which is 2

$2^3 = 2 \times 2 \times 2 = 8$

Powers of 10

To raise a number 10 to a power, it is enough to add after one a number of zeros equal to the exponent.

For example, let's raise the number 10 to the second power. First write down the number 10 itself and as an exponent write down the number 2

10^2

Now put an equal sign, write one and after this one write two zeros, because the number of zeros must be equal to the exponent

$10^2 = 100$

So the number 10 to the second power is the number 100. This is because the number 10 to the second power is the product of two factors, each of which is equal to 10

$10^2 = 10 \times 10 = 100$

Example 2. Let's raise the number 10 to the third power.

In this case there will be three zeros after the one:

$10^3 = 1000$

Example 3. Let's raise the number 10 to the fourth power.

In this case there will be four zeros after the one:

$10^4 = 10000$

Example 4. Let's raise the number 10 to the first power.

In this case there will be one zero after the one:

$10^1 = 10$

Numbers 10, 100, 1000 as a power with base 10

To represent the numbers 10, 100, 1000 and 10000 as a power with base 10, you need to write the base of 10, and as the exponent specify a number equal to the number of zeros of the original number.

Let's represent the number 10 as a power with base 10. We see that it has one zero. So the number 10 as a power with base 10 will be represented as 10^1

$10 = 10^1$

Example 2. Let us represent the number 100 as a power of base 10. We see that number 100 contains two zeros. Therefore, the number 100 as a power with base 10 is represented as 10^2

$100 = 10^2$

Example 3. Let's represent the number 1,000 as a power with base 10.

$1\ 000 = 10^3$

Example 4. Let's represent the number 10,000 as a power with base 10.

$10\ 000 = 10^4$

NEGATIVE NUMBERS AND EXPONENTS

If you raise a negative number to a power, you must put it in brackets.

For example, let's raise a negative number -2 to the second power. The number -2 to the second power is the product of two factors, each of which is equal to (-2)

$(-2)^2 = (-2) \times (-2) = 4$

If we didn't bracket the number -2, we would be calculating the expression -2^2, which is not equal to 4. The expression -2^2 would be -4. To understand why, let's touch on a few points.

When we put a minus in front of a positive number, we thereby perform the **operation of taking the opposite value**.

Suppose a number 2 is given and we need to find its opposite number. We know that the opposite of 2 is -2. To find the opposite of 2, we just need to put minus in front of the number. Inserting a minus in front of a number is already considered a full-fledged operation in mathematics. This operation, as mentioned above, is called the operation of taking the opposite value.

In the case of expression -2^2 there are two operations: the operation of taking the opposite value and raising to a power. Exponentiation is a higher priority operation than taking the opposite value.

Therefore, expression -2^2 is calculated in two steps. First, an exponentiation operation is performed. In this case, the positive number 2 was raised to the second power

Then the opposite value was taken. This opposite value was found for the value 4. And the opposite value for 4 is -4

$-2^2 = -4$

The brackets, on the other hand, have the highest execution priority. Therefore, in the case of calculating the expression $(-2)^2$, we first take the opposite value and then raise the negative number -2 to the second power. The result is positive 4, because the product of negative numbers is a positive number.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Example 2. Raise -2 to the third power.

-2 to the third power is the product of three factors, each of which is equal to (-2)

$(-2)^3 = (-2) \times (-2) \times (-2) = -8$

Example 3. Raise -2 to the fourth power.

The number -2 to the fourth power is the product of four factors, each of which is equal to (-2)

$(-2)^4 = (-2) \times (-2) \times (-2) \times (-2) = 16$

It is easy to see that raising a negative number to a power can result in either a positive or a negative answer. The sign of the answer depends on the exponent of the original power.

If the exponent is even, the answer is positive. If the exponent is odd, the answer will be negative. Let's show this using the number -3 as an example

$$(-3)^1 = (-3) = -3$$

$$(-3)^2 = (-3) \times (-3) = 9$$

$$(-3)^3 = (-3) \times (-3) \times (-3) = -27$$

$$(-3)^4 = (-3) \times (-3) \times (-3) \times (-3) = 81$$

In the first and third cases, the exponent was an **odd** number, so the answer became **negative**.

In the second and fourth cases, the exponent was an **even** number, so the answer became **positive**.

Example 7. Raise the number -5 to the third power.

The number -5 to the third power is the product of three factors each equal to -5. The exponent 3 is an odd number, so we can say in advance that the answer will be negative:

$(-5)^3 = (-5) \times (-5) \times (-5) = -125$

Example 8. Raise the number -4 to the fourth power.

The number -4 to the fourth power is the product of four factors, each of which is -4. The exponent of 4 is even, so we can say in advance that the answer will be positive:

$(-4)^4 = (-4) \times (-4) \times (-4) \times (-4) = 256$

FINDING VALUES OF EXPRESSIONS

When finding values of expressions that do not contain brackets, the exponentiation will be performed first, followed by multiplication and division in order, and then addition and subtraction in order.

Example 9. Find the value of the expression $2 + 5^2$

First we perform exponentiation. In this case, number 5 is raised to the second power to obtain 25. Then this result is added to the number 2

$2 + 5^2 = 2 + 25 = 27$

Example 10. Find the value of the expression $-6^2 \times (-12)$

First we perform exponentiation. Note that the number -6 is not bracketed, so number 6 will be raised to the second power, and then the result will be preceded by minus:

$-6^2 \times (-12) = -36 \times (-12)$

Complete the example by multiplying -36 by (-12)

$-6^2 \times (-12) = -36 \times (-12) = 432$

Example 11. Find the value of the expression -3×2^2

First we perform exponentiation. Then the result is multiplied with the number -3

$-3 \times 2^2 = -3 \times 4 = -12$

If the expression contains parentheses, you must first perform the actions in those brackets, then exponentiation, then multiplication and division, and then addition and subtraction.

Example 12. Find the value of the expression $(3^2 + 1 \times 3) - 15 + 5$

First perform the actions in brackets. Inside the parentheses, we apply the rules we learned earlier, i.e. first raise number 3 to the second power, then multiply 1 × 3, then add the results of raising number 3 to the power and multiplying 1 × 3. Subtraction and addition are then performed in order. Let's arrange this order of operations over the original expression:

$$\overset{1}{(}\,3^{2}\ \overset{3}{+}\ 1\ \overset{2}{\times}\ 3)\ \overset{4}{-}\ 15\ \overset{5}{+}\ 5$$

$(3^2 + 1 \times 3) - 15 + 5 = 12 - 15 + 5 = 2$

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Example 13. Find the value of the expression $2 \times 5^3 + 5 \times 2^3$

First, let's raise the numbers to a power, then perform multiplication and add up the results:

$2 \times 5^3 + 5 \times 2^3 = 2 \times 125 + 5 \times 8 = 250 + 40 = 290$

IDENTICAL TRANSFORMATIONS OF POWERS

Various identity transformations can be performed on powers, thereby simplifying them.

Suppose we need to calculate the expression $(2^3)^2$. In this example, two to the third power is raised to the second power. In other words, the power is raised to another power.

$(2^3)^2$ is the product of two powers, each of which is 2^3

$$2^3 \times 2^3$$

Each of these powers is the product of three factors, each of which is equal to 2

$$\underbrace{2 \times 2 \times 2}_{2^3} \times \underbrace{2 \times 2 \times 2}_{2^3}$$

We obtain the product $2 \times 2 \times 2 \times 2 \times 2 \times 2$, which is 64. Then the value of the expression $(2^3)^2$ or equals 64

$$(2^3)^2 = 2^3 \times 2^3 = 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 64$$

This example can be greatly simplified. To do this, the indices of expression $(2^3)^2$ can be multiplied and the product can be written over base 2

$$(2^3)^2 = 2^{3 \times 2} = 2^6$$

We got 2^6. Two to the sixth power is the product of six factors, each of which is 2. This product is 64

$$(2^3)^2 = 2^{3 \times 2} = 2^6 = 64$$

This property works because 2^3 is the product of $2 \times 2 \times 2$, which in turn is repeated twice. Then it turns out that the base 2 is repeated six times. Hence we can write that $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2$ is 2^6

In general, for any basis **a** with exponent **m** and **n**, the following equality holds:

$(a^n)^m = a^{n \times m}$

This identity transformation is called **raising the power to a power**. It can be read as follows: *"When raising the power to a power, the base is left unchanged and the exponents are multiplied"*.

After multiplying the exponents, you get another power, the value of which can be found.

Example 2. Find the value of the expression $(3^2)^2$

In this example, the base is 3 and the numbers 2 and 2 are exponents. Let's use the rule of raising to a power. Leave the base unchanged, and multiply the exponents:

$$(3^2)^2 = 3^{2\times 2} = 3^4$$

We got 3^4. And the number 3 to the fourth power is 81

$$(3^2)^2 = 3^{2\times 2} = 3^4 = 81$$

Consider the rest of the transformations.

MULTIPLYING POWERS

To multiply the powers, you need to calculate each power separately, and multiply the results.

For example, multiply 2^2 by 3^3.

2^2 is number 4 and 3^3 is number 27. Multiply the numbers 4 and 27 and you get 108

$2^2 \times 3^3 = 4 \times 27 = 108$

In this example, the bases of the powers were different. In case the bases are the same, you can write one base, and as the exponent write the sum of the exponents of the original powers.

For example, multiply 2^2 by 2^3

In this example, the bases of the powers are the same. In this case we can write down one base 2 and as an exponent write the sum of exponents of powers 2^2 and 2^3, i.e. we can leave the base unchanged, and add the exponents of the original powers. It will look like this:

$$2^2 \times 2^3 = 2^{2+3} = 2^5$$

We got 2^5. The number 2 to the fifth power is 32

$$2^2 \times 2^3 = 2^{2+3} = 2^5 = 32$$

This property works because 2^2 is the product of 2 × 2, and 2^3 is the product of 2 × 2 × 2. Then we obtain the product of five identical factors, each of which is equal to 2. This product can be represented as 2^5

$$2^2 \times 2^3 = \underbrace{2 \times 2 \times 2 \times 2 \times 2}_{2^5} = 2^5 = 32$$

In general, for any **a** and the exponents **m** and **n**, the following equality holds:

$$a^m \times a^n = a^{m+n}$$

This identity transformation is called the basic property of the power. It can be read as follows: *"When multiplying powers with equal bases, the base is left unchanged and the exponents are added"*.

Note that this transformation can be applied to any number of powers. The main thing is that the basis is the same.

For example, find the value of the expression $2^1 \times 2^2 \times 2^3$. Let's leave the base 2 unchanged, and add up the exponents:

$$2^1 \times 2^2 \times 2^3 = 2^{1+2+3} = 2^6 = 64$$

In some tasks, it is sufficient to perform the corresponding transformation without calculating the final power. This is, of course, very convenient, since calculating large powers is not easy.

Example 1. Present the expression $5^8 \times 25$ as a power

In this task we need to make one power instead of the expression $5^8 \times 25$

The number 25 can be represented as 5^2. Then we get the following expression:

$$5^8 \times 5^2$$

In this expression we can apply the main property of the power - leave the base 5 unchanged, and add the exponents 8 and 2:

$$5^8 \times 5^2 = 5^{8+2} = 5^{10}$$

The task can be considered solved because we have presented the expression $5^8 \times 25$ as one power, namely as the power of 5^{10}.

Let us write down the solution briefly:

$$5^8 \times 25 = 5^8 \times 5^2 = 5^{10}$$

Example 2. Present the expression $2^9 \times 32$ as a power

The number 32 can be represented as 2^5. Then we get the expression $2^9 \times 2^5$. Next, we can apply the base property of the power - leave base 2 unchanged, and add the exponents 9 and 5. The result is the following solution:

$$2^9 \times 32 = 2^9 \times 2^5 = 2^{14}$$

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Example 3. Calculate the product 3 × 3 using the basic property of a power.

Everyone knows very well that three times three equals nine, but the task requires you to use the basic property of the power in the solution. How do we do this?

Recall that if the number is given without an exponent, then the exponent must be considered equal to one. Therefore the factors 3 and 3 can be written as 3^1 and 3^1

$3^1 \times 3^1$

Now let's use the basic property of the power. We leave the base 3 unchanged, and add 1 and 1:

$3^1 \times 3^1 = 3^2$

Then we calculate the value of the expression. The number 3 to the second power is equal to 9

$3^1 \times 3^1 = 3^2 = 9$

Example 4. Calculate the product $2 \times 2 \times 3^2 \times 3^3$ using the basic property of powers.

Replace the product of 2 × 2 by $2^1 \times 2^1$, then by 2^{1+1}, and then by 2^2. Replace the product of $3^2 \times 3^3$ by 3^{2+3} and then by 3^5

$$2 \times 2 \times 3^2 \times 3^3 = 2^1 \times 2^1 \times 3^2 \times 3^3 = 2^{1+1} \times 3^{2+3} = 2^2 \times 3^5$$

Then we calculate the value of each power and find the product:

$$2 \times 2 \times 3^2 \times 3^3 = 2^1 \times 2^1 \times 3^2 \times 3^3 = 2^{1+1} \times 3^{2+3} = 2^2 \times 3^5 = 4 \times 243 = 972$$

Example 5. Perform the multiplication of x × x

These are two identical alphabetic factors with exponents of 1. For clarity, write down the exponents. Next, leave the base of x unchanged and add the exponents:

$$x \times x = x^1 \times x^1 = x^{1+1} = x^2$$

When answering a task like this at school, you should not write down the multiplication of powers with equal bases in such detail as is done here. Such calculations should be done in your head. The teacher is likely to be annoyed and give you a lower grade. Here, however, detailed notation is given so that the material will be as easy to understand as possible.

The solution to this example should preferably be written like this:

$$x \times x = x^2$$

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Example 6. Multiply $x^2 \times x$

The exponent of the second factor is one. Let us write it down for clarity. Next, leave the base unchanged, and add the exponents:

$$x^2 \times x = x^2 \times x^1 = x^{2+1} = x^3$$

Example 7. Multiply $y^3 \times y^2 \times y$

The exponent of the third factor is one. Let us write it down for clarity. Next, leave the base unchanged, and add the exponents:

$$y^3 \times y^2 \times y = y^3 \times y^2 \times y^1 = y^{3+2+1} = y^6$$

Example 8. Multiply $a \times a^3 \times a^2 \times a^5$

The exponent of the first factor is one. Let us write it down for clarity. Next, leave the base unchanged, and add the exponents:

$$aa^3a^2a^5 = a^1a^3a^2a^5 = a^{1+3+2+5} = a^{11}$$

Example 9. Present power 3^8 as a product of powers with equal bases.

In this task we have to make a product of powers whose bases will be 3 and whose sum of exponents will be 8. Any exponent can be used. Let us represent power 3^8 as a product of powers 3^5 and 3^3

$$3^8 = 3^5 \times 3^3$$

In this example we again relied on the basic property of the power. After all, the expression $3^5 \times 3^3$ can be written as 3^{5+3}, hence 3^8.

Of course, it was possible to represent power 3^8 as a product of other powers. For example, as $3^7 \times 3^1$, since this product also equals 3^8

$$3^8 = 3^7 \times 3^1$$

Representing a power as a product of powers with identical bases is mostly creative work. So you don't have to be afraid to experiment.

Example 10. Represent the power of x^{12} as different products of powers with bases x.

Let us use the basic property of power. Let's represent x^{12} as products with bases x, and sum of exponents equal to 12

$$x^{12} = x^{1+11} = x \times x^{11}$$

$$x^{12} = x^{2+10} = x^{2} \times x^{10}$$

$$x^{12} = x^{3+9} = x^{3} \times x^{9}$$

$$x^{12} = x^{1+1+10} = x \times x \times x^{10}$$

$$x^{12} = x^{1+1+2+8} = x \times x \times x^{2} \times x^{8}$$

The constructions with the sums of the exponents were written for clarity. Most often they can be omitted. Then you get a compact solution:

$$x^{12} = x \times x^{11}$$

$$x^{12} = x^{2} \times x^{10}$$

$$x^{12} = x^{3} \times x^{9}$$

$$x^{12} = x \times x \times x^{10}$$

$$x^{12} = x \times x \times x^{2} \times x^{8}$$

POWER OF A PRODUCT

In order to raise a product to a power, you must raise each factor of that product to a specified power and multiply the results.

For example, raise to the second power the product 2 × 3. Put this product in brackets and give 2 as the exponent

$$(2 \times 3)^{2}$$

Now raise each factor of the product 2 × 3 to the second power and multiply the results:

$$(2 \times 3)^{2} = 2^{2} \times 3^{2} = 4 \times 9 = 36$$

The way this rule works is based on the definition of power, which was given at the beginning.

To raise the product 2 × 3 to the second power is to repeat the product twice. And if you repeat it twice, you can get the following:

2 × 3 × 2 × 3

The product does not change from rearranging the places of the factors. This allows you to group the same factors together:

2 × 2 × 3 × 3

Repeating factors can be replaced by short entries - bases with exponents. The product 2 × 2 can be replaced by 2^2, and the product 3 × 3 can be replaced by 3^2. Then the expression 2 × 2 × 3 × 3 turns into $2^2 \times 3^2$.

Let **ab** be the original product. To raise this product to power **n**, we must separately raise the factors **a** and **b** to the specified power **n**

$$(ab)^n = a^n b^n$$

This property holds for any number of factors. The following expressions are also valid:

$$(abc)^n = a^n b^n c^n$$

$$(abcd)^n = a^n b^n c^n d^n$$

Example 2. Find the value of the expression $(2 \times 3 \times 4)^2$

In this example, the product 2 × 3 × 4 must be raised to the second power. To do this, take each factor of the product to the second power and multiply the results:

$$(2 \times 3 \times 4)^2 = 2^2 \times 3^2 \times 4^2 = 4 \times 9 \times 16 = 576$$

Example 3. To raise the product $a \times b \times c$ to the third power

Put this product in brackets, and as an exponent we give the number 3

$$(a \times b \times c)^3$$

Next, raise each factor of the product to the third power:

$$(a \times b \times c)^3 = a^3 \times b^3 \times c^3$$

---

example 4. Raise to the third power the product 3 × x × y × z

Let's bracket this product, and as an exponent let's specify 3

$(3 \times x \times y \times z)^3$

Let us raise each factor of the product to the third power:

$(3 \times x \times y \times z)^3 = 3^3x^3y^3z^3$

The number 3 to the third power is 27. Let's leave the rest unchanged:

$(3xyz)^3 = 3^3x^3y^3z^3 = 27x^3y^3z^3$

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

In some examples, multiplication of powers with equal exponent can be replaced by the product of bases with one exponent.

For example, calculate the value of the expression $5^2 \times 3^2$. Raise each number to the second power and multiply the results:

$5^2 \times 3^2 = 25 \times 9 = 225$

But it is not necessary to calculate each power separately. Instead, the given product of powers can be replaced by a product with one exponent $(5 \times 3)^2$. Then calculate the value in brackets and raise the result to the second power:

$5^2 \times 3^2 = (5 \times 3)^2 = (15)^2 = 225$

In this case again the rule of exponentiation of the product was used. If $(a \times b)^n = a^n \times b^n$, then $a^n \times b^n = (a \times b)^n$. That is, the left and right parts of the equality have swapped places.

POWER OF A POWER RULE (EXPONENTS)

We considered this transformation as an example when we tried to understand the essence of identical power transformations.

The base is left unchanged and the exponents are multiplied when the power is raised to a power:

$(a^n)^m = a^{n \times m}$

For example, the expression $(2^3)^2$ is an increase to power - two to the third power is raised to the second power. To find the value of this expression, the base can be left unchanged, and the exponents multiplied:

$(2^3)^2 = 2^{3 \times 2} = 2^6$

Then calculate the power of 2^6, which is 64

$(2^3)^2 = 2^{3 \times 2} = 2^6 = 64$

This rule is based on the previous rules: exponentiation of the product and the main property of the power.

Let us return to the expression $(2^3)^2$. The expression in brackets 2^3 is the product of three identical factors, each equal to 2. Then in expression $(2^3)^2$ the power inside the brackets can be replaced by the product 2 × 2 × 2.

$(2 \times 2 \times 2)^2$

And this is the exponentiation of the product that we studied earlier. Recall that to exponentiate a product, each factor of the product must be multiplied to the specified power and the results multiplied:

$(2 \times 2 \times 2)^2 = 2^2 \times 2^2 \times 2^2$

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Now we are dealing with the main property of the power. We leave the base unchanged, and add the exponents:

$(2 \times 2 \times 2)^2 = 2^2 \times 2^2 \times 2^2 = 2^{2+2+2} = 2^6$

As before we got 2^6. The value of this power is 64

$(2 \times 2 \times 2)^2 = 2^2 \times 2^2 \times 2^2 = 2^{2+2+2} = 2^6 = 64$

A product whose factors are also powers can also be raised to a power.

For example, find the value of the expression $(2^2 \times 3^2)^3$. Here, the exponents of each factor must be multiplied by the total exponent of 3. Then find the value of each power and calculate the product:

$(2^2 \times 3^2)^3 = 2^{2\times3} \times 3^{2\times3} = 2^6 \times 3^6 = 64 \times 729 = 46656$

The same thing happens when you raise a product to the power of a product. We said that in the power of a product, each factor of that product is raised to a specified power.

For example, to raise the product 2 × 4 to the third power, you must write the following expression:

$$(2 \times 4)^3 = 2^3 \times 4^3$$

But earlier it was said that if a number is given without an exponent, then the exponent should be considered equal to one. It turns out that the factors of the product 2 × 4 originally have exponent equal to 1. It means that the expression $2^1 \times 4^1$ was raised to the third power. This is power of a power.

Let's rewrite the solution using the rule of power of a power. We should get the same result:

$$(2^1 \times 4^1)^3 = 2^{1\times3} \times 4^{1\times3} = 2^3 \times 4^3$$

Example 2. Find the value of the expression $(3^3)^2$

We leave the base unchanged, and multiply the exponents:

$$(3^3)^2 = 3^{3\times2} = 3^6$$

We got 3^6. The number 3 to the sixth power is 729

$$(3^3)^2 = 3^{3\times2} = 3^6 = 729$$

Example 3. Perform exponentiation of the expression $(xy)^3$

Let's raise each factor of the product to the third power:

$$(xy)^3 = x^3y^3$$

Example 4. Perform exponentiation of the expression (abc)^5

Let's raise each factor of the product to the fifth power:

$$(abc)^5 = a^5b^5c^5$$

Example 5. Perform exponentiation of the expression $(-2ax)^3$

Let's raise each factor of the product to the third power:

$$(-2ax)^3 = (-2)^3a^3x^3$$

Since the negative number -2 was raised to the third power, it was bracketed.

Next you need to calculate what is being calculated. In this case you can calculate $(-2)^3$ to get -8. The letter part will remain unchanged:

$$(-2ax)^3 = (-2)^3a^3x^3 = -8a^3x^3$$

Example 6. Perform exponentiation of the expression $(10xy)^2$

$$(10xy)^2 = 10^2x^2y^2 = 100x^2y^2$$

Example 7. Perform exponentiation of the expression $(-5x)^3$

$$(-5x)^3 = (-5)^3x^3 = -125x^3$$

Example 8. Perform exponentiation of the expression $(-3y)^4$

$$(-3y)^4 = (-3)^4y^4 = 81y^4$$

Example 9. Perform exponentiation of the expression $(-2abx)^4$

$$(-2abx)^4 = (-2)^4a^4b^4x^4 = 16a^4b^4x^4$$

Example 10. Simplify the expression $x^5 \times (x^2)^3$

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

Let's leave the power of x^5 unchanged for now, and in the expression $(x^2)^3$ let's raise the power to a power:

$x^5 \times (x^2)^3 = x^5 \times x^{2 \times 3} = x^5 \times x^6$

Now let's multiply $x^5 \times x^6$. To do this, use the basic property of the power - leave the base of **x** unchanged, and add the exponents:

$x^5 \times (x^2)^3 = x^5 \times x^{2 \times 3} = x^5 \times x^6 = x^{5+6} = x^{11}$

Example 11. Find the value of the expression $4^3 \times 2^2$ using the main property of the power.

The basic property of powers can be used if the bases of the original powers are the same. In this example, the bases are different, so first we have to modify the original expression a bit, namely, to make the bases of the powers to be the same.

Let's take a close look at the power of 4^3. The base of this power is number 4, which can be represented as 2^2. Then the original expression will look like $(2^2)^3 \times 2^2$. If we raise the power of $(2^2)^3$ to a power, we obtain 2^6. Then the original expression will take the form $2^6 \times 2^2$, which can be calculated using the main property of the power.

Let us write down the solution of this example:

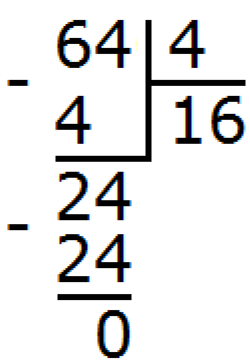
$$4^3 \times 2^2 = (2^2)^3 \times 2^2 = 2^6 \times 2^2 = 64 \times 4 = 256$$

DIVIDING POWER

To do division of powers, you need to find the value of each power, then do division of prime numbers.

For example, divide 4^3 by 2^2.

Calculate 4^3, we get 64. Calculate 2^2, we get 4. Now divide 64 by 4, we get 16



If the bases are the same when dividing the powers, the base can be left unchanged, and the exponent of the divisor can be subtracted from the exponent of the dividend.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

For example, find the value of the expression $2^3 : 2^2$

Let's leave the base 2 unchanged, and subtract the exponent of the divisor from the exponent of the dividend:

$$2^3 : 2^2 = 2^{3-2} = 2^1 = 2$$

So the value of the expression $2^3 : 2^2$ is 2.

This property is based on the multiplication of powers with equal bases, or as we used to say, on the basic property of the power.

Let's go back to the previous example $2^3 : 2^2$. Here the dividend is 2^3 and the divisor is 2^2.

To divide one number by another means to find a number that, when multiplied by the divisor, yields a resulting dividend.

In our case, to divide 2^3 by 2^2 is to find a power that, when multiplied by the divisor of 2^2, results in 2^3. And what power can be multiplied by 2^2 to get 2^3 ? Obviously, only the power of2^1. From the basic property of powers we have:

$$2^1 \times 2^2 = 2^{1+2} = 2^3$$

To make sure that the value of $2^3 : 2^2$ is 2^1, we can directly calculate the expression $2^3 : 2^2$. To do this, first find the value of the power of 2^3, we get 8. Then we find the value of the power of 2^2, we get 4. Divide 8 by 4 to get 2 or 2^1, because $2 = 2^1$.

$2^3 : 2^2 = 8 : 4 = 2$

Thus, when dividing powers with equal bases, the following equality is satisfied:

$$a^m : a^n = a^{m-n}$$

It may also happen that not only the bases are the same, but also the exponents. In this case the answer will be one.

For example, find the value of the expression $2^2 : 2^2$. Calculate the value of each power and divide the resulting numbers:

$$2^2 : 2^2 = 4 : 4 = 1$$

When solving example $2^2 : 2^2$, you can also apply the rule of division of powers with equal bases. The result is a number to the power of zero, because the difference between the exponents of powers 2^2 and 2^2 is zero:

$$2^2 : 2^2 = 2^{2-2} = 2^0$$

In mathematics, it is accepted that any number to the power of zero is one:

$$2^2 : 2^2 = 2^{2-2} = 2^0 = 1$$

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

We found out above why the number 2 to the power of zero equals one. If you calculate 22 : 22 by the usual method, without using the rule of division of powers, you will get one.

Example 2. Find the value of the expression $4^{12} : 4^{10}$

Use the rule of division of powers. Let's leave the base 4 unchanged, and subtract the exponent of the divisor from the exponent of the dividend:

$4^{12} : 4^{10} = 4^{12-10} = 4^2 = 16$

Example 3. Present the quotient $x^3 : x$ as a power with base x

Use the rule of division of powers. We leave the base of x unchanged, and subtract the exponent of the divisor from the exponent of the dividend. The exponent of the divisor is one. For clarity, write it down:

$$x^3 : x = x^3 : x^1 = x^{3-1} = x^2$$

Example 4. Present the quotient $x^3 : x^2$ as a power with base x

Let us use the rule of division of powers. Let's leave the base of x unchanged, and subtract the exponent of the divisor from the exponent of the dividend:

$$x^3 : x^2 = x^{3-2} = x^1 = x$$

The division of powers can be written as a fraction. Thus, the previous example can be written as follows:

$$\frac{x^3}{x^2} = x^{3-2} = x$$

The numerator and denominator of a fraction $\frac{x^3}{x^2}$ may be written in expanded form, namely, as products of identical factors. The power of x^3 can be written as **x × x × x**, and the power of x^2 as **x × x**. Then the construction x^{3-2} can be skipped and the fraction can be shortened. The numerator and denominator can be reduced by two x factors each. The result will be one factor x

$$\frac{x^3}{x^2} = \frac{x \times \overset{1}{\cancel{x}} \times \overset{1}{\cancel{x}}}{\underset{1}{\cancel{x}} \times \underset{1}{\cancel{x}}} = \frac{x \times 1 \times 1}{1 \times 1} = \frac{x}{1} = x$$

Or even shorter:

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

$$\frac{x^3}{x^2} = \frac{x \times \overset{1}{\cancel{x}} \times \overset{1}{\cancel{x}}}{\underset{1}{\cancel{x}} \times \underset{1}{\cancel{x}}} = x$$

Also, it is useful to be able to quickly reduce fractions consisting of powers. For example, a fraction $\frac{x^3}{x^2}$ can be reduced by x^2. To reduce $\frac{x^3}{x^2}$ by x^2, divide the numerator and denominator of $\frac{x^3}{x^2}$ by x^2

$$\frac{x^3}{x^2} = \frac{x^3 : x^2}{x^2 : x^2} = \frac{x^{3-2}}{x^{2-2}} = \frac{x^1}{x^0} = \frac{x}{1} = x$$

The division of powers may not be described in detail. The given reduction can be made shorter:

$$\frac{x^3}{x^2} = \frac{x^3 : x^2}{x^2 : x^2} = \frac{x}{1} = x$$

Or even shorter:

$$\frac{\cancel{x^3}^{\,x}}{\underset{1}{\cancel{x^2}}} = x$$

Example 5. Perform division $x^{12} : x^3$

Let us use the rule of division of powers. Let's leave the base of x unchanged, and subtract the exponent of the divisor from the exponent of the dividend:

$$x^{12} : x^3 = x^{12-3} = x^9$$

Write down the solution using fraction reduction. Let us write the division of powers $x^{12} : x^3$ as $\frac{x^{12}}{x^3}$. Then let us reduce this fraction by x^3.

$$\frac{x^{12}}{x^3} = \frac{x^{12} : x^3}{x^3 : x^3} = \frac{x^9}{x^0} = \frac{x^9}{1} = x^9$$

Example 6. Find the value of the expression $\frac{7^9 \times 7^5}{7^{12}}$

In the numerator, multiply powers with equal bases:

$$\frac{7^9 \times 7^5}{7^{12}} = \frac{7^{9+5}}{7^{12}} = \frac{7^{14}}{7^{12}}$$

Now apply the rule of division of powers with equal bases. We leave the base 7 unchanged, and subtract the exponent of the divisor from the exponent of the dividend:

$$\frac{7^9 \times 7^5}{7^{12}} = \frac{7^{9+5}}{7^{12}} = \frac{7^{14}}{7^{12}} = 7^{14-12} = 7^2$$

Complete the example by calculating the power of 7^2

$$\frac{7^9 \times 7^5}{7^{12}} = \frac{7^{9+5}}{7^{12}} = \frac{7^{14}}{7^{12}} = 7^{14-12} = 7^2 = 49$$

Example 7. Find the value of the expression $\frac{2^5 \times (2^3)^4}{2^{13}}$

In the numerator, raise the power to a power. Do this with expression $(2^3)^4$

$$\frac{2^5 \times (2^3)^4}{2^{13}} = \frac{2^5 \times 2^{3\times4}}{2^{13}} = \frac{2^5 \times 2^{12}}{2^{13}}$$

Now do the multiplication of powers with equal bases in the numerator:

$$\frac{2^5 \times (2^3)^4}{2^{13}} = \frac{2^5 \times 2^{3\times4}}{2^{13}} = \frac{2^5 \times 2^{12}}{2^{13}} = \frac{2^{5+12}}{2^{13}} = \frac{2^{17}}{2^{13}}$$

Now apply the rule of division of powers with equal bases:

$$\frac{2^5 \times (2^3)^4}{2^{13}} = \frac{2^5 \times 2^{3\times4}}{2^{13}} = \frac{2^5 \times 2^{12}}{2^{13}} = \frac{2^{5+12}}{2^{13}} = \frac{2^{17}}{2^{13}} = 2^{17-13} = 2^4 = 16$$

So the value of expression $\frac{2^5 \times (2^3)^4}{2^{13}}$ is 16

In some examples, it is possible to reduce the same factors in the course of the solution. This simplifies the expression and the calculation as a whole.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

For example, find the value of expression $\frac{4^3 \times 3^2}{2^6}$. Let us write the power of 4^3 as the power of $(2^2)^3$. Then we obtain the following expression:

$$\frac{4^3 \times 3^2}{2^6} = \frac{(2^2)^3 \times 3^2}{2^6}$$

In the numerator, perform the exponentiation of the power. Do this with expression $(2^2)^3$

$$\frac{4^3 \times 3^2}{2^6} = \frac{(2^2)^3 \times 3^2}{2^6} = \frac{2^6 \times 3^2}{2^6}$$

The numerator and denominator of the resulting expression contain the power 2^6, which can be reduced by 2^6

$$\frac{4^3 \times 3^2}{2^6} = \frac{(2^2)^3 \times 3^2}{2^6} = \frac{\overset{1}{\cancel{2^6}} \times 3^2}{\underset{1}{\cancel{2^6}}} = 3^2 = 9$$

We see that this leaves us with the only power of 3^2, the value of which is 9.

Example 8. Find the value of the expression $\frac{28^6}{7^5 \times 4^5}$

The denominator is the product of powers with the same exponent. According to the rule of exponentiation of the product, the construction $7^5 \times 4^5$ can be represented as a power with one exponent $(7 \times 4)^5$. Next, multiply the expression in brackets to get 28^5. As a result, the original expression will take the following form:

$$\frac{28^6}{28^5}$$

Now you can apply the rule of division of powers:

$$\frac{28^6}{28^5} = 28^{6-5} = 28^1 = 28$$

So, the value of expression $\frac{28^6}{7^5 \times 4^5}$ is 28. Let's write down the solution in full:

$$\frac{28^6}{7^5 \times 4^5} = \frac{28^6}{(7 \times 4)^5} = \frac{28^6}{28^5} = 28^{6-5} = 28^1 = 28$$

EXPONENTS OF FRACTIONS

To raise a fraction to a power, you must raise the numerator and denominator of the fraction to a specified power.

For example, raise fraction $\frac{2}{3}$ to the second power. Put this fraction in brackets and give 2 as the exponent

$$\left(\frac{2}{3}\right)^2$$

If we do not bracket the whole fraction, it is equivalent to raising only the numerator of the fraction. If we want to raise the fraction $\frac{2}{3}$ to the second power, we should not write it as $\frac{2^2}{3}$.

So, to calculate the value of expression $\left(\frac{2}{3}\right)^2$, you need to raise the numerator and denominator of this fraction to the second power:

$$\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2}$$

We obtained a fraction with powers in the numerator and denominator. Calculate each power separately

$$\left(\frac{2}{3}\right)^2 = \frac{2^2}{3^2} = \frac{4}{9}$$

So the fraction $\frac{2}{3}$ to the second power is equal to the fraction $\frac{4}{9}$.

The above rule works as follows. The fraction $\frac{2}{3}$ to the second power is the product of two fractions, each of which is equal to $\frac{2}{3}$

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \times \frac{2}{3}$$

We remember that to multiply fractions, you must multiply their numerators and denominators:

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2}{3 \times 3}$$

And since the numerator and denominator are multiplied by the same factors, the expressions 2 × 2 and 3 × 3 can be replaced by 2^2 and 3^2, respectively:

$$\left(\frac{2}{3}\right)^2 = \frac{2}{3} \times \frac{2}{3} = \frac{2 \times 2}{3 \times 3} = \frac{2^2}{3^2}$$

From which you get the answer $\frac{4}{9}$.

In general, for any **a** and **b ≠ 0** the following equality holds:

$$\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$$

This identity transformation is called **exponentiation of a fraction**.

Example 2. To raise fraction $\frac{3}{5}$ to the third power

We conclude this fraction in brackets and give number 3 as the exponent. Then raise the numerator and denominator of the fraction to the third power and calculate the resulting fraction:

$$\left(\frac{3}{5}\right)^3 = \frac{3^3}{5^3} = \frac{27}{125}$$

A negative fraction is raised to a power in the same way, but before calculating you should decide what sign the answer will have. If the exponent is even, the answer will be positive. If the exponent is odd, the answer will be negative.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

For example, raise fraction $-\frac{1}{2}$ to the second power:

$$\left(-\frac{1}{2}\right)^2$$

The exponent is an even number. So the answer will be positive. Then we apply the rule of raising a fraction to a power and calculate the resulting fraction:

$$\left(-\frac{1}{2}\right)^2 = \frac{1^2}{2^2} = \frac{1}{4}$$

The answer is positive because $\left(-\frac{1}{2}\right)^2$ is the product of two factors, each equal to a fraction of $-\frac{1}{2}$

$$\left(-\frac{1}{2}\right)^2 = -\frac{1}{2} \times \left(-\frac{1}{2}\right)$$

And the product of negative numbers (including rational numbers) is a positive number:

$$\left(-\frac{1}{2}\right)^2 = -\frac{1}{2} \times \left(-\frac{1}{2}\right) = \frac{1}{4}$$

If the fraction $-\frac{1}{2}$ is raised to the third power, the answer will be negative, because in this case the exponent will be an odd number. The rule of exponentiation remains the same, but before performing this exponentiation, we will need to put a minus:

$$\left(-\frac{1}{2}\right)^3 = -\frac{1^3}{2^3} = -\frac{1}{8}$$

Here the answer is negative because expression $\left(-\frac{1}{2}\right)^3$ is the product of three factors, each of which is a fraction of $-\frac{1}{2}$.

CREDIT: WWTL STUDENTS (PKCE, ACCE, AMCE)

$$\left(-\frac{1}{2}\right)^3 = -\frac{1}{2} \times \left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right)$$

First we multiplied $-\frac{1}{2}$ and $-\frac{1}{2}$ and got $\frac{1}{4}$, but then we multiplied $\frac{1}{4}$ by $-\frac{1}{2}$ and got a negative answer of $-\frac{1}{8}$

$$\left(-\frac{1}{2}\right)^3 = \underbrace{-\frac{1}{2} \times \left(-\frac{1}{2}\right)}_{\frac{1}{4}} \times \left(-\frac{1}{2}\right) = \frac{1}{4} \times \left(-\frac{1}{2}\right) = -\frac{1}{8}$$

Example 3. Find the value of the expression $\left(\frac{2}{4}\right)^2 - \frac{3}{16}$

Let's perform exponentiation of fractions:

$$\left(\frac{2}{4}\right)^2 - \frac{3}{16} = \frac{2^2}{4^2} - \frac{3}{16} = \frac{4}{16} - \frac{3}{16}$$

Then we calculate the value of the resulting expression:

$$\left(\frac{2}{4}\right)^2 - \frac{3}{16} = \frac{2^2}{4^2} - \frac{3}{16} = \frac{4}{16} - \frac{3}{16} = \frac{1}{16}$$

EXPONENTS OF DECIMALS

When you raise a decimal to the power, you must put it in brackets. For example, let's raise a decimal 1.5 to the second power

$$(1.5)^2 = 1.5 \times 1.5 = 2.25$$

It is allowed to convert a decimal to a fraction and to exponentiate that fraction. Solve the previous example by converting a decimal to a fraction:

$$(1.5)^2 = \left(\frac{15}{10}\right)^2 = \frac{15^2}{10^2} = \frac{225}{100} = 2.25$$

Example 2. Find the value of the power of $(-1.5)^3$

The exponent is an odd number. So the answer will be negative

$$(-1.5)^3 = (-1.5) \times (-1.5) \times (-1.5) = -3.375$$

Example 3. Find the value of the power of $(-2.4)^2$

The exponent is an even number. So the answer will be positive:

$$(-2.4)^2 = (-2.4) \times (-2.4) = 5.76$$